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SENSE AND BASIC LAW V IN FREGE'S LOGICISM*

In his lecture "Funktion und Begriff" of 1891, Frege introduced the notion of sense and his Basic Law V. This fact, that Frege's two main innovations since the publication of the *Begriffsschrift* in 1879 were introduced simultaneously, reflects important features of the development of Frege's logicist project. On the same occasion, Frege also stated that the expressions flanking the principal identity sign of Basic Law V have the same sense. I argue, *pace* Dummett and the received reading of Frege's semantics, that this view on Basic Law V is the principal reason why Frege introduced sense. My reasoning is based on the observation that some time after 1879, Frege realized that in order to carry out his logicist project he needed a formal means of licencing a transformation from functions to their extensions or courses-of-values. But then, I argue, Frege had to solve two pressing tasks. The first is that the explanation of the identity relation in *Begriffsschrift* is incompatible with the assumption that Basic Law V is a primitive logical law. Frege therefore needed an alternative account of the identity relation. The second is that a distinction must be drawn between logically primitive identity statements and derivable ones. I claim that the notion of sense ought to be seen as the key to Frege's response to these challenges. By virtue of it, he could reinterpret the identity relation, and a consequence of the claim that the expressions flanking the identity sign of Basic Law V have the same sense is that the law is self-evidently true, and thus a non-reducible logical truth. In a crucial sense, Basic Law V differs sharply from arithmetical identity statements, in that such statements have different senses on the two sides of the identity sign. They are therefore not self-evident, but must be proved.

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1. The Topic

Frege introduced his two-tiered semantics of sense and reference in *Funktion und Begriff* (1891). On the same occasion he made a claim that has led to some controversy in the secondary literature, namely that the two sides of Basic Law V (Frege 1893) have the same sense. Some scholars, notably Sluga (1980, pp. 147–157), argue that the fact that this claim is made *simultaneously* with the introduction of the notion of sense, reflects significant features of Frege's way of individualizing that notion. Other interpreters have followed the proposal of Dummett (1991b), and maintain that the claim is completely out of order with Frege's more elaborated treatment of sense (as in Frege 1892b, 1893 and 1918); they take Frege to have made an unfortunate aberration in his first announcement of his new semantics.

Dummett's influential view on the individualization of sense and thought (the sense expressed by a sentence) is captured by his *principle K*:

> If one sentence involves a concept that another sentence does not involve, the two sentences cannot express the same thought or have the same content. (Dummett 1991b, p. 295)

In this paper I argue, *pace* Dummett, that until he received Russell's famous letter of June 16, 1902, Frege held the viewpoints made in *Funktion und Begriff* (1891). My reading of Frege's notion of sense is evidently inconsistent with *principle K*, and I attempt to establish that, as a consequence of fundamental features of his logicist project, Frege never could have accepted *principle K*, or any somewhat similar principles (cf. e.g. Bell 1987 and Beaney 1996 for proposals of principles for individualizing sense highly inspired by Dummett's reading of Frege).

In my view, we ought to read Frege's writings of 1879–1903 against the background of the program of logicism.[1] This paper reflects this attitude by connecting the introduction of the notion of sense to the profound changes in Frege's philosophy of logic that took place when Basic Law V was added to the basic laws of the *Begriffsschrift* (Frege 1879).[2] The main significance of this paper might be its assertion that one cannot give an adequate account of Frege's notion of sense

[1] I have previously argued (Alnes 1998) that Frege's writings after he gave up logicism are also mainly directed at issues in the philosophy of logic and mathematics.

[2] The term "philosophy of logic" is used for convenience only. Frege would not have accepted *our* distinction between "logic" and "philosophy of logic" (cf. e.g. Alnes 1998, Goldfarb 1979, Ricketts 1986, Weiner 1990).

unless one has realized that it belongs at the very heart of Fregean logicism. The manner in which I connect the introduction of sense to Frege's justification of logicism makes the present line of reading differ from that of Sluga (1980), as well as from any other reading I have encountered in the secondary literature.

To be more specific, the purpose of this paper is to establish that Frege introduced his new semantics as a solution to a task he found himself facing some time after the publication of Frege 1879, *viz.* that of drawing the crucial line between two different kinds of *analytic* identity statements: those that are *self-evident* [*einleuchtend/selbstverständlich*] or logically *primitive*, like Basic Law V, and those that are not self-evident and therefore must be proved. This distinction, as we shall see, plays a major role in the philosophical foundation of Frege's program of proving that arithmetic is an analytic science.

My reasoning in favor of a polymorphous relationship between sentence and thought consists of several steps. In the two sections that follow, the necessary historical background is outlined: Some relevant aspects of the development of Fregean logicism from 1879 to 1893 are introduced (Section 2), and Frege's explications or elucidations of the epistemic status of the basic laws of logic of Frege 1879 are scrutinized (Section 3). Then I turn to the development of Frege's view on identity (Section 4). I connect this theme to Frege's method of forming purely logical proper names in the Begriffsschrift, and I give, in addition to the systematic points, some exegetic observations against *principle K* (Section 5). During these discussions we obtain the framework needed for insight into the intimate connection between the introduction of sense and the adding of Basic Law V to the Begriffsschrift. I argue that the claim that the two sides of Basic Law V express the same sense is meant to vindicate it as a basic logical law, and not a derivable sentence (Section 6). The paper concludes with some remarks on the famous misgivings about Basic Law V that Frege articulated *before* he received Russell's letter. Dummett has taken these remarks to support *principle K*, but I argue that they fit well into the interpretation outlined in this paper (Section 7).

2. Basic Law V and the Development of Frege's Logicism

Frege delivered the talk "Funktion und Begriff" at the meeting of the "Jenaischen Gesellschaft für Medicin und Naturwissenschaft" on January 9, 1891 and published it as a monograph the same year. To the publication Frege added a preface, where it is stated that:

> In the near future ... it is my intention to give an exposition of how I express in my Begriffsschrift[3] the fundamental definitions of arithmetic and of how, starting from these, I construct proofs that make exclusive use of my own signs. For that purpose I shall find it useful to be able to refer to this lecture rather than be drawn, in the course of my exposition, into discussions which some might regard as necessary but others as reprehensibly irrelevant to the main point. (Frege 1891, p. i)

The future work alluded to is *Grundgesetze der Arithmetik* (Frege 1893). Clearly, this prospective comment shows that the theme of the talk is closely related to the program of proving that arithmetic can be reduced to logic. The observation that Frege's motivation for publishing the lecture is the logicist project has a notable bearing on how I locate the sense-reference distinction within Frege's overall philosophy.

As observed at the beginning of this paper, in his talk Frege presented, in quasi-formal terms, the unfortunate Basic Law V, and he introduced the distinction between sense and reference. For the record I begin with a brief discussion of Basic Law V, which, in a quasi-modern notation, looks like this:

$$\vdash (\grave{\varepsilon}f(\varepsilon) = \grave{\alpha}g(\alpha)) \mathbf{=} (\forall x)(f(x) \mathbf{=} g(x)).^4$$

The sign "$\vdash$" is composed of the horizontal stroke, or the "content stroke", and the vertical stroke, or the "judgment stroke". Since the basic laws and theorems of the Begriffsschrift are preceded by this sign, they are asserted, and this, in turn, means that the Begriffsschrift is a language and not a formal notation as this word is used in our contemporary jargon.[5] The symbols "$\grave{\varepsilon}f(\varepsilon)$" and "$\grave{\alpha}g(\alpha)$" represent

[3]When referring to the language that Frege developed in Frege 1879, and refined in Frege 1893, 1903, I use "Begriffsschrift" rather than "conceptual notation" or "concept-script".

[4]In order to simplify the discussion, and make it more accessible to the reader of today, I use a version of standard contemporary logical notation. When quoting from Frege's writings, I have allowed myself to make the appropriate notational changes.

For reasons that cannot be treated in any detail here, but are alluded to in note 12, Frege never distinguished between identity and equivalence. In Frege 1879, the sign for identity is "$\equiv$", while in his works of 1893 and 1903, it is "$=$". The purpose of the use of bold identity signs is to indicate where we would use a sign for equivalence rather than the identity sign. (This method of representation might be more useful with respect to Basic Law V$'$, discussed in Section 7 below.) Frege's understanding of the horizontal stroke changed when he decided to take the reference of sentences to be full-fledged objects (compare Frege 1879, §2 with Frege 1893, p. X and §5).

[5]For an account of the fundamental contrast between Frege's conceptualization of logic, where a separation into model theory and proof theory is excluded, and the contemporary one which is based on this separation, cf. for instance Alnes 1998,

the courses-of-values of functions "$f()$" and "$g()$", and a course-of-value, as Frege views matters, is an object. The notion of a course-of-value is a generalization of that of an extension, and we might think of the course-of-value of a function as a set of ordered pairs, such that its first member is an argument and its second member is the value of the function for that argument. We might, for instance, think of the course-of-value of the function "x^2" as the infinite set: $\{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,4\rangle, \ldots, \langle \odot, 0\rangle, \ldots\}$;[6] or, to use another picture, we might think of it as a graph (cf. Frege 1891, p. 9). Basic Law V states that if two functions have the same courses-of-values, then they take the same value for each and every argument, and *vice versa.* An alternative explanation of the law would be to say that it asserts that for any two functions, "$f()$" and "$g()$", either side of the principal identity-sign always refers to the same object, the True or the False.

Frege added Basic Law V to his Begriffsschrift because he had realized that in order to define expressions that refer to the (cardinal) numbers in the Begriffsschrift, he needed a formal means of licencing a purely logical transition from functions to their extensions. In his talk, Frege claimed that the right-hand side of Basic Law V expresses the same sense as its left-hand side, "but in a different way. It presents the sense as an equality holding generally; whereas the newly-introduced expression [that of courses-of-values] is simply an equation" (Frege 1891, p. 11). This accentuation of the difference between the two sides of the law will be important in our line of reasoning.

Frege's first elaborated discussion of the theory of sense and reference is in Frege 1892b.[7] He seeks to justify the distinction by way of several examples framed in the vernacular language, and by discerning so-called "propositional attitude ascriptions". This might seem to go against the claim that sense is introduced mainly due to questions that arise in connection with his logicism. But it does not. For obviously, unless all signs, or maybe better, all significant linguistic unities, of all languages express sense, the introduction of that notion with respect to the Begriffsschrift would certainly have been *ad hoc.* Therefore, the treatment of more or less familiar and intuitively clear cases is needed in order to justify a notion that has a decisive role to play elsewhere, *viz.* at the very core of Fregean logicism.

Goldfarb 1979, van Heijenoort 1967b, Hintikka 1986 and Resnik 1986. Tappenden (1997) is critical of what he calls the orthodoxy of the Harvard school. His critical points are challenging, but must be left for another occasion.

[6]Here the sign "$\odot$" stands for the sun (cf. Frege 1891, p. 19 f.).

[7]In Frege 1984, "Über Sinn und Bedeutung" is translated as "On Sense and Meaning". For reasons that I cannot go into in this paper (but see Alnes 1998), I prefer the original translation as "On Sense and Reference".

For the overall argument of this paper, it is of decisive importance to be aware of the fact that the derivations and proofs made in the third chapter of Frege 1879, "Some Topics from a General Theory of Sequences", are exclusively devoted to the logic of the theory of sequences (in our contemporary jargon, we could say that the chapter deals with the theory of binary relations). Frege saw clearly that the first and crucial step of logicism is to prove that mathematical induction is a purely logical notion, and, accordingly, neither an exclusively mathematical principle, nor a principle that needs an appeal to intuition for its justification.[8] Frege focused on what we could call "the second step of logicism", that of defining logical proper names that refer to logical objects, some time after he had finished Frege 1879. This change of focus is clear from the observation that the major aim of *Grundlagen der Arithmetik* (Frege 1884) is to establish both that numbers are full-fledged objects and that it is possible to refer to them by means of pure logic.

3. On the Epistemic Status of the Basic Laws of the Begriffsschrift

In this Section we take a brief look at Frege's delineation of the basic laws in Frege 1879. This early manner of circumscribing the basic laws of logic will be contrasted to the more complex treatment needed when Basic Law V entered the scene. In the first chapter of Frege 1879, "Exposition of the Symbols",[9] the primitive signs of the Begriffsschrift are presented, and in the second chapter, "Representation and Derivation of some Judgments of Pure Thought", the nine basic laws

[8]Cf. Frege 1879, §§24 and 26, where a generalized version of mathematical induction is defined. Some illuminating discussions about the results achieved in Frege 1879, chapter III, can be found in Boolos 1985 and Demopoulos 1994. In particular, Demopoulos argues that "The major conceptual innovation of [*Begriffsschrift*, chapter] III is the definition of the ancestral, and the principal result is the proof of its connectedness" (Demopoulos 1994, p. 235). Cf. also Alnes 1998 for an elaborate discussion about Frege's reasons for focusing on the theory of sequences before attempting to define the individual numbers in the Begriffsschrift. This remark, from Frege 1893, §9, reflects the present point: "The introduction of a notation for courses-of-values seems to me to be one of the most important supplementations that I have made of my *Begriffsschrift* since my first publication on this subject." For a rich and informative, albeit controversial, history about the attempts at avoiding an appeal to intuitions in semantics, cf. Coffa 1991.

[9]The title of the chapter is "Erklärung der Bezeichnungen". The translation in van Heijenoort 1967a, as "Definition of the Symbols", is unfortunate, since there are no definitions in that chapter, only elucidations of the symbols (cf. Frege 1879, §13). On the crucial distinction between statements proper and definitions on the one hand, and elucidations on the other, cf. Alnes 1998 and Weiner 1990.

are formulated. Since it will provide useful when discussing the basic laws, I begin by quoting Frege's way of introducing the conditionality sign, the identity sign and the generality sign.

The conditionality sign is presented thus:

> If A and B stand for contents that can become judgments (§2), there are the following four possibilities:
>
> (1) A is affirmed and B is affirmed;
>
> (2) A is affirmed and B is denied;
>
> (3) A is denied and B is affirmed;
>
> (4) A is denied and B is denied.
>
> Now
>
> $$\vdash B \supset A$$
>
> stands for the judgment that *the third of these possibilities does not take place, but one of the three others does* (Frege 1879, §5).

Then we turn to the identity sign:

> Now let
>
> $$\vdash B \equiv A$$
>
> mean that *the sign A and the sign B have the same conceptual content, so that we can everywhere put B for A and conversely.* (Frege 1879, §8)

The detectable difference between the two kinds of explications, in that the conditional sign is given an objectual reading while the identity sign is given a metalinguistic reading, is an issue for the next Section.

The generality sign, i.e. what after Russell is called "the universal quantifier", is introduced thus:

$$\vdash (\forall x)(\Phi(x)),$$

> ... *stands for the judgment that, whatever we may take for its argument, the function is a fact.* (Frege 1879, §11)

Let us turn to Frege's discussion of his basic laws; I have chosen, as illustrative examples, one of the three basic laws that are based on the conditionality sign, one of the three basic laws that are based on the identity sign, and the one that is based on the generality sign. The first basic law of conditionality is "$\vdash a \supset (b \supset a)$", which Frege paraphrases as: "The case in which a is denied, b is affirmed, and a is affirmed is excluded" (Frege 1879, §14). "This is evident [*Dies leuchtet ein*]", Frege says, "since a cannot at the same time be denied and affirmed". Here Frege appeals to the exposition of the conditionality sign and the principle of contradiction: one cannot both affirm and deny one and the same expression for a thought. This is why the first basic law is self-evident [*einleuchtend*]. This is the first law of identity: "$\vdash (c \equiv d) \supset f(c) \supset f(d)$", which Frege paraphrases as: "The case in which the content of c is identical with the content of d and in which $f(c)$ is affirmed and $f(d)$ is denied does not take place" (Frege 1879, §20). This law is self-evident because to try to deny it would be to try to say both that c is identical to d and that $f(c)$ and not $f(d)$, and the latter, according to Frege's explication of the identity sign, is the same as to affirm and deny that $f(c)$. Thus, anybody who understands the identity sign immediately realizes that this law is true; thus it is self-evident. And finally, we have the law of generality: "$\vdash (\forall x)(f(x) \supset f(c))$". This is Frege's comment: "$(\forall x)(f(x))$ means that $f(x)$ takes place, whatever we may understand by x. If therefore $(\forall x)(f(x))$ is affirmed, $f(c)$ cannot be denied" (Frege 1879, §22). Again, as is clear from the exposition of the generality sign, to try to deny the law would be to try to affirm and deny simultaneously that $f(c)$, which is impossible, since it would go against the principle of contradiction. The basic law of universality is therefore self-evident.[10] Expositions corresponding to these can easily be given for each of the six remaining basic laws of Frege 1879; thus they are all self-evident.[11]

We shall see that a principal reason for Frege to introduce sense is that a notion of self-evidence as essentially based on the principle of contradiction is inadequate when Basic Law V is added to the set of *primitive* laws of logic. The notion of sense enabled Frege to invoke a more sophisticated notion of self-evidence.

4. Conceptual Content and Identity

Until he introduced the sense-reference distinction, Frege relied on a unary semantics, that of "conceptual content". The conceptual content of an expression is individuated in terms of its role in inferences: two expressions, A and B, have the same conceptual content if the con-

[10]Evidently, in his early writings, Frege held a traditional view on the notion of self-evidence, since his elucidations of the basic laws of the Begriffsschrift presuppose the principle of contradiction. One should keep in mind, however, that these elucidations in their turn presuppose the Begriffsschrift, and, consequently, that it is at least partly by virtue of this language, where the classical view on propositions as consisting of a subject and a predicate has been replaced by an argument-function approach, that one immediately realizes that the basic laws are self-evident.

[11]However, as shown in Łukasiewicz 1936, the "Third Fundamental Law of Conditionality" can be proved from the first and second laws of conditionality, and is thus not independent.

sequence that follows from A when it is combined with certain other expressions, always follows from B when it is combined with the same expressions, and *vice versa* (cf. Frege 1879, §3). On the basis of an objectual reading of the identity relation, this allows the well-known puzzle that if two proper names (including sentences) single out the same object (or fact),[12] then they have the same conceptual content, and all identity statements seem to be entirely trivial. As he found this consequence intolerable, Frege opted for the metalinguistic reading of the identity sign presented in the former Section. This is his more elaborate account:

> Identity of content differs from conditionality and negation in that it applies to names and not to contents ... [The signs] suddenly display their own selves when they are combined by means of the sign for identity of content; for it expresses the circumstance that two names have the same content. Hence the bifurcation of a sign for identity of content necessarily produces a bifurcation in the meaning of all signs: they stand at times for their content, at times for themselves. (Frege 1879, §8)

Now, given this explanation, do we really need two (or several) proper names with the same conceptual content in the Begriffsschrift at all?—that is to say, is the identity sign needed in the Begriffsschrift? After all, an intuitive criterion for a language to be a Begriffsschrift—a "perfect language"—is that it approximates to the highest possible degree the ideal of a one-to-one correspondence between sign and content. (Of course, Frege was fully aware that due to a conventional choice of variables, and alternative complete sets of primitive logical constants, the ideal can never be fully realized.) I believe Frege's answer to our question to be this: the sign is needed because the Begriffsschrift has a pivotal role to play in the sciences and in the representation of the structure of human knowledge (cf. Frege 1879, p. VI). Or, to formulate the point in our contemporary jargon, the identity sign is needed when

[12]Cf. Frege 1879, §3, where "Umstand", "Satz", and "Thatsache" are used for the content of a sentence. There arise some interpretative complications here, since it is not clear whether Frege at the time thought of the conceptual content of a sentence (i.e. its "judgeable content") as a full-fledged object. In Frege 1879, only a sentence is allowed to follow what he then called "the content-stroke" (cf. Frege 1879, §2). In Frege 1891, 1893, any saturated expression might be preceded by what Frege, at that time, had re-named as "the horizontal" (cf. Frege 1893, p. X and §2). These issues are too broad to be discussed in any detail in this paper, but cf. Alnes 1998. One should note, however, that if the present essay is on the right track, then the connections between Frege's introduction of the notion of sense, his reinterpretation of the identity relation as objectual, and the specification of the reference of sentences as objects, are well worth some reflections. For instance, this view on the reference of sentences, if coherent, seems to require a two-tiered semantics.

the Begriffsschrift is applied to a field of study and thus extended into a theory. I shall defend this interpretation by looking at some further passages from Frege 1879, §8.

The passage quoted above continues:

> At first we have the impression that what we are dealing with pertains merely to the *expression* and *not to the thought*, that we do not need different signs at all for the same content and hence no sign whatsoever for identity of content. (Frege 1879, §8)

This is what we would expect given our earlier discussion, and Frege's next task is accordingly to demonstrate "that this is an empty illusion". In order to achieve this result, he invokes a geometrical example by presenting two ways of determining a given point, by intuition [*Anschauung*] and calculation, respectively, and then he explains:

> To each of these ways of determining the point there corresponds a particular name. Hence the need for a sign for identity of content rests upon the following consideration: the same content can be completely determined in different ways; but that in a particular case *two ways of determining it* really yield the *same result* is the content of a *judgment*. (Frege 1879, §8)

On this basis the following conclusion is drawn:

> The existence of different names for the same content is *not always merely an irrelevant question of form*; rather, that there are such names is the very heart of the matter if each is associated with a different way of determining the content. In that case the judgment that has identity of content as its object is *synthetic*, in the Kantian sense. (italics added)

Although generally overlooked in the secondary literature, it is evident from these passages that Frege had important reasons for illustrating his point by way of a *synthetic* example.[13] The implication of Frege's repeated distinction between "irrelevant questions of form", on the one hand, and "expression of thought", on the other, is that since purely logical judgements are analytic, different expressions for the same thought in the pure Begriffsschrift concern form only, and this means that the identity sign is needed with respect to synthetic identity judgments only. My point, then, is that in 1879, Frege thought that judgments that have the identity sign as their principal sign are not needed in pure logic, but are unavoidable when extra-logical vocabulary and extra-logical basic laws are added to the Begriffsschrift,

[13]In opposition to Russell, Frege followed Kant in taking the propositions of geometry to be synthetic *a priori*; cf. for instance Frege 1874, p. 1, 1879, p. VI and 1884, §14.

or when one wants, due to our limited survey abilities, to introduce, by way of definitions, abbreviations.[14]

This account of identity statements, however, cannot be upheld when Basic Law V is added as a primitive law to those originally given in Frege 1879. Basic Law V, like the other basic laws, is taken to be analytic (otherwise, Frege could not have attempted to prove that arithmetic is an analytic science), but still, the difference between the two sides of the identity sign cannot, as was observed in Section 2 above, be said to be just "an irrelevant question of form". The expression on the right-hand side consists of the identity sign, one second-order function name and two first-order function names, while the expression on the left-hand side is simply an ordinary identity and thus consists of nothing but the identity sign and two proper names.

To complete our present discussion, one crucial question needs to be raised. We have noted that the unary semantics of conceptual content, combined with the assumption that an identity statement should be given an objectual reading, has the unfortunate consequence that one cannot draw the distinction between identity statements that are solely a "question of form", and thus devoid of logical significance, and identity statements that have a further significance. (Hereafter I sometimes abbreviate the distinction between the types of statements as "trivial" and "nontrivial", respectively.)[15] But, one might reasonably ask, could not Frege have modified his metalinguistic analysis in such a way that it could serve as an account of the nontrivial, but purely logical, identity statements?[16] As a matter of fact, Frege tried this way out, but failed to reach a satisfying analysis. Before treating this issue, I shall put forth Frege's reasons for giving up the metalinguistic analysis of identity.

This is the famous opening passage of Frege 1892b:

Equality gives rise to challenging questions which are not altogether easy to answer. Is it a relation? A relation between objects, or between names or

[14]Frege's view on the significance of definitions at this early stage of his philosophical development is clear: "A more extrinsic reason for the introduction of a sign for identity of content is that it is at times expedient to introduce an abbreviation for a lengthy expression. Then we must express the identity of content that obtains between the abbreviation and the original form (Frege 1879, §8)." Of course, the limited survey abilities of human beings are logically irrelevant, and thus, Frege thought at the time, definitions are introduced for psychological reasons only.

Since our concern here is Frege's early understanding of the identity relation, we will not discuss the development of Frege's views on definition; but see for instance Alnes 1998, Dummett 1991a, Tappenden 1995, and Weiner 1984, 1990 for somewhat diverging treatments of this huge and complex topic.

[15]"Trivial" and "nontrivial" given the Begriffsschrift (cf. note 10).

[16]I am thankful to the referee, Linnebo, for posing this important question.

signs of objects? In my *Begriffsschrift* I assumed the latter. The reasons which seem to favor this are the following: $a = a$ and $a = b$ are obviously statements of differing cognitive value [*Erkenntniswerte*];[17] $a = a$ holds *a priori* and according to Kant, is to be labeled analytic, while statements of the form $a = b$ often contain very valuable extensions of our knowledge and cannot always be established *a priori*. (Frege 1892b, p. 25)[18]

Here Frege repeats the view of Frege 1879 by drawing a distinction between two kinds of statements of the form $a = b$, those that are analytic and those that are not, and he suggests that those that are not analytic often extend our knowledge. He continues by presenting the solution to the problem raised in Frege 1879 (in the first sentence of the passage) and then he argues that it fails (in the rest of the passage):

What we apparently want to state by $a = b$ is that the signs or names 'a' and 'b' designate the same thing, so that those signs themselves would be under discussion; a relation between them would be asserted. But this relation would hold between the names or signs only in so far as they named or designated something. It would be mediated by the connection between each of the two signs with the same thing. But this is arbitrary. Nobody can be forbidden to use any arbitrarily producible event or object as a sign for something. In that case the sentence $a = b$ would no longer refer to the subject matter, but only to its mode of designation; we would express no proper knowledge by its means. (Frege 1892b, p. 26)

What kinds of reflections could have lead Frege to this argument? This question is pressing, since we seek to find the motivation behind the introduction of sense to arise from considerations concerning the program of proving that arithmetic is reducible to logic; or alternatively, the program of proving that arithmetical judgments can be expressed as judgments of pure logic.[19] The following proposal might seem to be a reasonable answer to our question.

[17]For a discussion of Frege's term "Erkenntniswert", and the widespread misreading of it in our contemporary philosophy of language, cf. Alnes 1998.

[18]Note the surprising dialectic that Frege starts by asking whether identity is a relation, and then continues to discuss what kind of relation it is; he seems to take the answer to the opening question as a given. (This observation was made by John Perry in a lecture in Oslo, in the spring of 1998.)

[19]Our question ought to be seen as problematic also for those who take the discussion in Frege 1879 and the introduction of sense to be motivated by reflections over propositional attitude ascriptions. To take just one striking example, Nathan Salmon has, on this basis, argued that among the objections to a metalinguistic reading of identity statements, that of Frege 1892b:

. . . must surely be seen as one of the weakest. The objection constitutes a great irony in Frege's philosophy. Frege claims that the fact that two names—say "Hesperus" and "Phosphorus"—happen to name the same thing is an uninteresting accident of the use of language, a result of arbitrary linguistic convention, and is irrelevant to the subject matter—in this case, astronomy-determined by the object so named, whereas the fact that the objects Hesperus and Phosphorus are the same thing is an

The adding of Basic Law V to the Begriffsschrift made Frege realize that the difference between 'a' and 'b' in an *analytic* judgment of the form $a = b$ might, after all, go beyond an "irrelevant question of form". The quotation offered above indicates that Frege had come to the conclusion that to take the identity relation to be metalinguistic has the devastating consequence that an identity statement relates the quasi-empirical (or syntactical) objects that flank the identity sign.[20] Since the signs, as stated in the passage under consideration, are arbitrary, *and*, according to the metalinguistic reading, the judgment that "$2+2 = 4$" is the judgment that "$2+2$" and "4" have the same content, this judgment depends on our name-giving practice; it depends on empirical features of our system of numerals, and it therefore seems to be synthetic after all. Our system of numerals could, for instance, have been such that the numeral "4" referred to the number 3. When Frege came to conclude that the metalinguistic analysis was deeply confused and could not serve his aims, he felt himself left with one option, that of explaining nontrivial identity statements on the basis of an objectual reading.[21]

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interesting fact of astronomy and is independent of human decision or convention. What Frege failed to notice is that this claim is categorically denied by the very theory of sense that he used the argument to motivate. (Salmon 1986, p. 52)

And on the preceding page, Salmon maintains that:

> By the time he came to write "Über Sinn und Bedeutung," Frege found reason to reject this analysis [that of Frege 1879] of identity statements. This is all for the good, since the "Begriffsschrift" account, taken as an analysis of natural-language identity statements, is surely mistaken ... central elements of Frege's Puzzle have nothing special to do with the identity relation ... The "Begriffsschrift" account is thus not only mistaken but irrelevant. (Salmon 1986, p. 51)

The ascription to Frege (and to him in particular!) of such simple confusions is a strong signal that something has gone wrong. In this case, it is the assumption that Frege ought to be read as a semanticist mainly concerned with certain limited topics in the philosophy of language.

[20] I owe the present reconstruction of Frege's reasoning to Linnebo.

[21] I am, of course, not claiming that Frege was right in giving up the metalinguistic reading of the identity sign; to give an analysis of the peculiar identity relation (that is, if there is such a relation), is obviously beyond our task here.

It might be noted, however, that the claim that signs are arbitrary seems to be a consequence of the introduction of sense. By now signs are merely conventional means to get access to what really counts, namely senses and thoughts. In Frege 1879, 1880–1881, 1882, on the other hand, the significance of signs is heavily stressed, and in these writings Frege underscores his debt to Leibniz, citing the presentation of Leibniz's logical ideas given in Trendelenburg 1867. In this work, Trendelenburg emphasizes the pivotal role Leibniz gave to signs. It therefore seems clear that before he introduced sense, Frege would have objected to any claim to the effect that signs are arbitrary.

Let us return to the question of when Frege settled for an objectual reading of the identity sign. In the preface to Frege 1893 we encounter this telling retrospective reflection on the development of his logical thinking:

> With this book I carry out a design that I had in view as early as my *Begriffsschrift* of 1879 and announced in my *Grundlagen der Arithmetik* of 1884. I wish here to substantiate in actual practice the view of Number [*Anzahl*] that I expounded in the latter book ... One reason why the execution appears so long after the announcement is to be found in internal changes in my *Begriffsschrift*, which forced me to discard an almost completed manuscript. These improvements may be mentioned here briefly. The primitive signs used in *Begriffsschrift* occur here also, with one exception. Instead of the three parallel lines I have adopted the ordinary sign of equality, since I have persuaded myself that it has in arithmetic precisely the meaning [*Bedutung*] that I wish to symbolize ... the opposition that may arise against this will very likely rest on an inadequate distinction between sign and thing signified. (Frege 1893, p. IX)

Frege's suggestion, in other words, is that the attempt at formalizing the informal analysis of Frege 1884 led him to change his view on the identity relation, and made him introduce the notion of sense. He even emphasizes the crucial distinction between use and mention relied on in the opening passage of Frege 1892b. Clearly, Frege began the attempt at formalizing the analysis of Frege 1884 soon after that book was completed, and, as the phrase "an almost completed manuscript" indicates, Frege worked on this first attempt for quite a while. We might therefore assume that he began to reformulate his logical views in terms of the sense–reference distinction some time between the late 1880s and 1891. As the passage furthermore makes evident, this fundamental change in Frege's thinking is intimately connected with his new view on the identity relation.[22]

The sense of a sign is that "wherein the mode of presentation is contained" (Frege 1892b, p. 26), and this is the solution to the puzzle about identity:

> If the sign 'a' is distinguished from the sign 'b' only as an object (here by means of its shape), not as a sign (i.e. not by the manner in which it designates something), the cognitive value [*Erkenntniswert*] of $a = a$ becomes essentially

[22] There are some obvious connections between the remark made here and the fact that in some of the formulas of Frege 1879, e.g. the crucial formula 115, which defines the notion of a *singled value procedure*, the quantifiers, due to the metalinguistic reading of the identity sign, are ambiguous in that they simultaneously range over both objects and signs of objects. If Frege ever became aware of this ambiguity, it would probably be as a result of his reflections on use and mention. (It is, by the way, no accident that in Frege 1879, the phrase "singled valued procedure" is used in Part III, while "function" is used in Part I; cf. Demopoulos 1994.)

equal to that of $a = b$, provided $a = b$ is true. A difference can only arise if the difference between the signs corresponds to a difference in the mode of presentation of the thing designated. (Frege 1892b, p. 26)

Here "sign" is any proper name (including, in Frege's extended use of the word, both indexicals, definite descriptions, and even sentences) of any language. By motivating the introduction of the notion of sense thus, Frege takes himself to have pointed out a general phenomenon pertaining to all languages. Since the notion of sense has been given an independent and general justification, it can safely be appealed to within the program of logicism.

To reiterate, in this Section I have sought to establish two aspects of the development of Frege's thinking: The first is that Frege fathomed the full force of the fact that purely logical identity judgments are needed in order to carry out the program of logicism, some time between the writings of Frege 1879 and Frege 1884; the second is that after he began to formalize the viewpoints of Frege 1884, he realized that the analysis of identity advocated in Frege 1879 is incapable of handling nontrivial, but logical, identity statements. Frege therefore reexamined the identity relation in terms of the new semantics of sense and reference.

In the next Section I spell out the essential polymorphous conceptualization of the sign-sense relationship that grew out of Frege's reflections over the means needed to form logical proper names in the Begriffsschrift. Thereafter, in the final two Sections, I return to Frege's reflections on the epistemic status of Basic Law V.

5. Logical Proper Names and "Carving Up Content"

Let us consider some of the passages under the heading "To obtain the concept of Number [*Anzahl*], we must fix the sense of a numerical identity" (Frege 1884, §§62–70). Frege formulated the aim of this central part of his analysis thus:

Our aim is to construct the content of a judgement which can be taken as an identity such that each side of it is a number. We are therefore proposing not to define identity specially for this case, but to use the concept of identity, taken as already known, as a means for arriving at that which is to be regarded as being identical. (Frege 1884, §63)

In order to avoid being accused of introducing an *ad hoc* method restricted to the aim at hand, Frege invoked a familiar (at least in his time—cf. footnote 23 below) geometrical example:

The judgement "line a is parallel to line b", or, using symbols, $a//b$, can be taken as an identity. If we do this, we obtain the concept of direction, and say: "the direction of line a is identical with the direction of line b". Thus we replace the symbol // by the more generic symbol =, through removing what is specific in the content of the former and dividing it between a and b. (Frege 1884, §64)

Immediately after this remark, we encounter the famous statement that "[w]e carve up the content in a way different from the original way, and this yields us a new concept". I will argue that by this statement, Frege meant that the method he relied on in these parts of Frege 1884 enables us to obtain a new concept without changing the underlying thought.

Frege specified the method used in order to achieve his aim in a letter to Russell written after he had introduced the notion of sense:

We can also try the following expedient, and I hinted at this in my *Grundlagen der Arithmetik*. If we have a relation $\Phi(\xi, \zeta)$ for which the following propositions hold: (1) from $\Phi(a, b)$ we can infer $\Phi(b, a)$, and from $\Phi(a, b)$ and $\Phi(b, c)$ we can infer $\Phi(a, c)$ then this relation can be transformed into an equality (identity), and $\Phi(a, b)$ can be replaced by writing e.g., $\S a = \S b$. (Frege 1980, p. 141)

Obviously, the first sentence points to Frege 1884, §§61–70. The idea is to take an equivalence relation as a starting point (in the characterization, Frege leaves out symmetry, i.e. $\Phi(a, a)$—the letter, after all, is to Russell) and turn it into an identity.[23] For the record, note that in the general case the new concept (in the present case represented by "$\S\xi$") might not be known in advance. (In note 32, I discuss one aspect of the significance of this feature of *some* of the transformations.) A passage in an undated letter to Peano, written some time after Frege 1903, is rather conclusive for the issues at stake:

Or we can proceed as follows: every line determines a class of equally long lines; if two lines are equally long, the two classes coincide; and this again

[23] For an informative discussion concerning where Frege got this idea, and its significance for his logicism, with respect to both historical and systematical considerations, cf. Wilson 1992. Wilson points out that the mentioned transformation from an equivalence relation of parallelism into an identity among directions of lines is taken from von Staudt, who, in turn, used the transformation as a device to refer to points at infinity, this as part of his attempt at founding extended Euclidian geometry. (The aim was to avoid any appeal to Poncelet's somewhat vague "principle of continuity", by carrying out the extension by logical principles instead.) Frege's affinity to von Staudt's ideas is particularly clear in the very first piece written by Frege: "'Point at infinity' ... designates the fact that parallel lines behave projectively like straight lines passing through the same point. 'Point at infinity' is therefore only another expression for what is common to all parallels, which is what we commonly call 'direction"'(Frege 1873, p. 3). Dummett 1991a provides a detailed and informative discussion of all of Frege 1884, and in particular of §§55–70, but he does not take the historical antecedents to the critical method of Frege's entire logicism into consideration.

gives us an identity. Let A and B be lines and let "$A \cong B$" say that these lines are congruent. Then, in my notation, $\grave{\varepsilon}(\varepsilon \cong A)$ is the class of lines equal in length to A, and

$$\grave{\varepsilon}(\varepsilon \cong A) = \grave{\varepsilon}(\varepsilon \cong B)$$

is the essential content [*wesentlicher Inhalt*] of "$A \cong B$" in the form of an equation. In similar cases, too, we can proceed in this way and express agreement in a certain respect in the form of an equation ... Of course, this has not yet explained how it is possible that identity should have a higher cognitive value [*Erkenntniswert*] than a mere instance of the principle of identity. In the proposition, "The evening star is the same as the evening star" we have only the latter; but in the proposition, "The evening star is the same as the morning star" we have something more. (Frege 1980, p. 127)

I think it is quite clear that when he uses "essential content", Frege is thinking of sense. This is for the following reason. The familiar examples of "The evening star is the same as the evening star" and "The evening star is the same as the morning star" are introduced with the purpose of demonstrating the generality of a puzzle introduced earlier in the letter, namely that any attempt at explicating the identity relation, as used in arithmetic, must take into consideration the fact that "$233 + 798 = 1031$" seems to say more, or to have a richer content [*Inhalt*], than "$1031 = 1031$". And Frege is now going to give the solution; the passage continues:

How can the substitution of one proper name for another designating the same heavenly body effect such changes? One would think that it could only affect the form and not the content. And yet anyone can see that the thought of the second proposition is different, and in particular that it is essentially richer in content than that of the former. This would not be possible if the difference between the two propositions resided only in the names "evening star" and "morning star", without a difference in content being somewhat connected with it. Now, both names designate the same heavenly body ... so the difference cannot lie in this.

Thus, the solution to the puzzle consists in the fact that the difference in content goes beyond reference. When Frege said that the two expressions mentioned in the first passage preserves the "essential content", the word "content" evidently had the same significance as here. Furthermore, Frege's next step is to label this notion of content "sense" and to distinguish it from "reference" (cf. *loc. cit.*). From all this we conclude that in the actual context, the word "content" is used synonymously with "sense", and this means that the transformation from "$A \cong B$"to "$\grave{\varepsilon}(\varepsilon \cong A) = \grave{\varepsilon}(\varepsilon \cong B)$" preserves more than just truth-value; it preserves sense.[24]

[24] In the previous note, it was observed that Frege obtained the transformation mentioned here from von Staudt, and, as Wilson 1992 has convincingly pointed out, von Staudt thought that the transformation preserves content. Wilson furthermore

In order to obtain the generalization needed for my interpretation, I shall investigate further the notion of transformation that Frege relied on in the discussed cases. We begin with a passage from the unpublished article "The Argument for my stricter Canons of Definition" from about 1897–98. The starting point is the following formula of Peano:

$$u, v \varepsilon K . f \varepsilon v \mathrm{f} u . \bar{f} \varepsilon u \mathrm{f} v . \supset \text{num } u = \text{num } v.$$

(Here "u" and "v" stand for classes, and the formula states that if f is a function that maps u into v, and its inverse function, $\bar{f}$ maps v into u, then the same number belongs to v and u.) In his devastating refutation of Peano's symbolization of generality, Frege invoked the rule of contraposition. About this rule he observed that:

> This transformation, called ... "contraposition" ... is indispensable. The sense is scarcely [*kaum*] affected by it, since the sentence gives neither more nor less information after the transformation than before. (Frege 1897–1898, p. 154)[25]

This remark hardly makes sense unless Frege held contraposition to be a transformation that preserves sense. Another case is that of double negation. Late in his career Frege stated that "'not (not B)' has the same sense as 'B'" (Frege 1923, p. 44).[26]

In his justification of the transformation from "not [A and B]" to "not [B and A]", Frege summarized the general principle that the transformations we have inspected are supposed to obey:

> This interchangeability should no more be regarded as a theorem here than for compound thoughts of the first kind [i.e. from "A and B" to "B and A"], for there is no difference in sense between these expressions. It is therefore self-evident [*Selbstverständlich*] that the sense of the second compound sentence is true if that of the first is true—for it is the same sense. (Frege 1923, p. 40 f.)

claims, as I do, that to Frege the transformation codified by Basic Law V preserves the underlying thought (cf. Wilson 1992, p. 169).

[25] The reason this transformation shows that Peano's formula cannot be right is that after the transformation, u does not stand for a class anymore. Thus, when the transformation has been carried out the sign has changed reference and, accordingly, sense. Frege's novel invention, of course, is to do the transformation *within* the scope of the quantifiers.

[26] Or to take another of Frege's examples: "The prefix 'un' is not always used to negate. There is hardly any difference in sense between 'unhappy' and 'miserable'. Here we have a word which is the opposite of 'happy', and yet not its negation. For this reason the sentences 'This man is not unhappy' and 'This man is happy' do not have the same sense" (Frege 1897, p. 150). By implication, if "unhappy" had been the negation of "happy", then "This man is not unhappy" and "This man is happy" would have expressed the same thought. This shows that double negation preserves sense.

I take this to mean that some purely logical transformations of sentences are such that the newly obtained sentence expresses the same thought as the old one, and, furthermore, that it is self-evident that such is the case. Double negation and contraposition are examples of such transformations, as is also the transformation of an equivalence relation into an identity based on a concept not presented by the equivalence relation itself. Common to all these cases, as is evident from the passage just quoted, is the fact that these logical transformations are not *proofs*, and thus no inferences are involved.

It is time to contrast the present reading, that Frege had an essentially polymorphous view on the sign-sense relation, with Dummett's *principle K* (cf. Section 1, above). It is clear that nearly all the pairs of sentences that we have looked at go against this principle; double negation and contraposition introduce the negation sign, which involves the concept of negation, into the new sentence, and the method of transforming an equivalence relation into an identity relation is more radical still; sometimes this way of "carving up content" even "yields us a new concept". Dummett mentions five pairs of sentences that Frege claimed express the same content or sense, but which go against *principle K* (in Dummett 1991b, p. 292 f.):

A1: a is parallel to b.

A2: The direction of $a =$ the direction of b.

B1: For every a $f(a) = g(a)$.

B2: The value-range of $f =$ the value-range of g.

C1: There are just as many Fs as Gs.

C2: The number of Fs is the same as the number of Gs.

D1: There exist unicorns.

D2: The number of unicorns is not nought.

E1: Jupiter has four moons.

E2: The number of Jupiter's moons is four.

We have looked at pairs A1–A2 and B1–B2. These two pairs have a pivotal role to play in Frege's logicism, and this is obviously also the case with respect to pair C1–C2, which, in Dummett's words, "is of course that pair with which Frege is really concerned in §§63–69 of *Grundlagen der Arithmetik*, expressed without his technical jargon" (*loc. cit.*). The role of pairs D1–D2 and E1–E2 is different, however. Frege mentioned them in order to exemplify the supposition that when it comes to the question of identifying a thought, the vernacular language is a deficient guide. This since, due to inherent psychological and pragmatic features, sentences of the ordinary languages often belie their logical structure.[27] Frege's point is that when the two sentences in each pair are *translated* into the Begriffsschrift, we *discover* that they are represented by the same formula, and thereby we realize that they express the same thought (cf. Frege 1892a, pp. 197–200). Since this involves a somewhat different issue than the one we are presently concerned with, let us leave it aside.[28] With respect to the pairs in focus, it should be clear that an unpleasant consequence of Dummett's reading, and of any reading which ascribes to Frege a principle for individuating thoughts somewhat similar to that of *principle K*, is that Frege was exceedingly confused with respect to his own logicism, because such readings entail that Frege was inconsistent at a very deep level; he claimed that some specific pairs of sentences express the same sense (or content), while his own theoretical framework committed him to the view that they do not have the same sense (content). Moreover, the invoked pairs of sentences lay at the very heart of Frege's logicist project.

When Dummett faces the question of the individualization of sense, it is taken for granted that the notion is introduced in order to give what Dummett has labeled "a theory of meaning". The present alternative approach, which I shall seek to establish in the final two Sections, enables us to avoid the unfortunate conclusion that Frege was confused in the simple way entailed by Dummett's interpretation.

6. Basic Law V and Sense

In opposition to the other basic laws, Basic Law V cannot be shown to hold by way of an elucidation that invokes the principle of contradiction and, eventually, the relation of identity. From the very outset Frege was aware that Basic Law V differed fundamentally from the rest of his basic laws in that an attempt to deny it does not necessitate both affirming and denying a thought as expressed by *one particular sentence*; i.e. a denial of Basic Law V does not lead to a judgement of the form *a and not a* (cf. Section 3, above). In his focal lecture, Frege stated that:

[27] Cf. e.g. Frege 1879, §3, 1880–1881, pp. 6 f. and 12 f., and 1897, p. 142.

[28] Our observation is important with respect to the broader issue concerning the principles for individuating sense, however. In accordance with the discussion of this paper, I would, in addressing this issue fully, emphasize the significance of the Fregean logicism and the role of the Begriffsschrift, rather than focus on contemporary themes in the philosophy of language, which, if I am correct, are more or less foreign to Frege's way of thinking.

> The possibility of regarding the equality holding generally between values of functions as a particular equality, *viz.* an equality between value-ranges is, I think, indemonstrable; it must be taken to be a fundamental law of logic. (Frege 1891, p. 10)

This is where the real significance of the transformations we have looked at comes to light. We have seen that a criterion for any law of logic, including Basic Law V, to be a *primitive* law of logic is that it is self-evident. Therefore, as with respect to the other transformations we have considered, Basic Law V cannot be demonstrated, i.e. proved, since to demand a proof here would be to maintain that it does not express the same sense on both sides of the identity sign. My claim, in other words, is that at the time Frege thought that Basic Law V codifies a purely logical transformation—an equality holding generally has been transformed into an identity—of signs that preserves an underlying thought. The structural similarity between this transformation and the transforming of an equivalence relation into an identity should be obvious. With regard to Basic Law V, then, a grasp of its sense suffices to establish its truth, and a mark of a primitive law of logic is that it expresses a self-evidently true thought. Furthermore, as was argued in the preceding Section, no proof or demonstration can establish that two expressions have the same sense. Note that the judgment that Basic Law V is true stands in sharp opposition to the judgement that the evening star is the morning star—here empirical knowledge is needed, and a grasp of expressed sense is accordingly insufficient to establish its truth-value.

In the lecture that motivated this paper, Frege argued that the expressions flanking the identity sign of an arithmetical identity statement of the form $a = b$, like “$2 + 2 = 4$”, do not have the same sense. (We encountered the same standpoint in a letter that Frege wrote to Peano more than ten years after he held the lecture.) Like “The evening star is the same as the morning star”, such identity statements are not self-evident (cf. 1891, p. 13 f.), but unlike “The evening star is the same as the morning star”, they are not empirical. The means that *proofs* must be furnished before the claim that these identities are analytic, and not synthetic, is justified.[29]

[29]Implicit in my reconstruction of Frege’s thinking is the assumption that a proof preserves reference, but not sense. This is evident from the following passage: “[T]he only criticism that can justly be made against this book concerns not the rigor but merely the choice of the course of proof and the intermediate steps. Frequently several routes for a proof are open; I have not tried to travel them all, and thus it is possible—even probable—that I have not invariably chosen the shortest.” (Frege 1893, p. VI f.) The very possibility of giving alternative proofs, of involving different

I have argued that to Frege, some analytic judgments of the form $a = b$ express the same sense on either side of the identity sign, namely the primitive identity statements of the Begriffsschrift;[30] some analytic judgements of the form $a = b$ do not express the same sense on either of the identity sign, namely those that must be proved; and all synthetic judgments of the form $a = b$ express different senses on either side of the identity sign. (We are here, of course, considering judgements whose content is perspicuously represented in the Begriffsschrift.)

To sum up the present discussion, my view is that a significant motivation behind Frege's introduction of the technical notion of sense is that he needed an apparatus that would enable him to draw these, rather subtle, distinctions between kinds of identity statements; with respect to Fregean logicism, the crucial distinction is, of course, that between kinds of analytic identity statements.

In order to substantiate further the line of thought ascribed to Frege, the final Section is devoted to some passages where Frege expressed misgivings about Basic Law V. I shall try to show that a noteworthy advantage of the framework outlined so far is that it enables us both to take Frege's claim about the sense expressed on either side of Basic Law V that underlies this essay seriously, and to provide a reasonable explanation of the troublesome second thoughts that Frege entertained about Basic Law V.

7. Frege's Second Thoughts

In the introduction to Frege 1893, it is admitted that:

> If anyone should find anything defective [concerning whether "the chains of inferences are cohesive and the buttresses solid"], he must be able to state precisely where, according to him the error lies: in the Basic Laws, in the Definitions, in the Rules, or in the application of the rules at a definite point. If we find everything in order, then we have accurate knowledge of the grounds upon which each individual theorem is based. A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values (V) ... I hold that it is a law of pure logic. In any event the place is pointed out where the decision must be made. (Frege 1893, p. VII)

Dummett takes this passage as evidence for his claim that by 1893, Frege had already realized that the viewpoints expressed in the focal

basic laws and definitions, shows that a proof cannot be taken to preserve sense. For in that case, the proved expression of a thought would express different thoughts according to the given proof, which is obviously absurd to Frege.

[30]This is also the case with respect to judgments obtained from definitions. But, as mentioned in note 14, Frege's understanding of definitions must be left for another occasion.

lecture, held no more that two years earlier, are inadequate: How could Frege, Dummett reasons, possibly have doubted Basic Law V, given that either side of the identity sign expresses the same sense? (Cf. Dummett 1991b, p. 293.) In opposition to Dummett, I shall seek to establish that Frege did not voice any doubts about the *truth* of Basic Law V in this passage, but rather made a much weaker concession, namely that he was uncertain whether the proposed basic law is a *primitive* law of logic, or whether, like "$2 + 2 = 4$", it ought to be provable in the Begriffsschrift.

As I view matters, Frege's vacillation on this issue is due to two problematic contrasts between Basic Law V and the rest of his basic laws. The first is that Basic Law V cannot be elucidated or explicated in the same manner as the other basic laws; the second is that Basic Law V has a considerably more complex structure than the other, by far, simpler basic laws. My view is that Frege was not entirely convinced that the expressions flanking the identity sign of Basic Law V do in fact express the same sense. Seen in the light of the earlier argument of this paper, this means that I do not think that Frege, until he received Russell's letter, doubted that the proposed basic law was true, but could not entirely convince himself that it was self-evidently true.

My reading is borne out in Frege 1903, §§147 and 148, where he admitted to being troubled by the fact that Basic Law V is of another nature than the rest of the basic laws, but all the same he maintained that it explicates a rule that the mathematicians implicitly had relied on for a long time. This claim was also made ten years earlier, in Frege 1893:

> This transformation [i.e. the one licenced by Basic Law V] must be regarded as a law of logic, a law that is invariably employed, even if tacitly, whenever discourse is carried on about the extensions of concepts. The whole Leibniz–Boole calculus of logic rests upon it. (Frege 1893, §9)

In the same Section, Frege insisted that "[t]he conversion of the generality of an identity into an identity of courses-of-values has to be capable of being carried out in our symbolism".

This makes it evident that until he received Russell's letter, Frege was persuaded that "[w]ithout such a means [i.e. a law that permits a transformation from an equality holding generally into an equation] a scientific foundation for arithmetic would be impossible" (1903, §47); and he was convinced that arithmetic *could be* given a "scientific foundation", i.e. could be founded on logic. But, all the same, he was not content with basing this scientific foundation on such a complex law as his fifth, and he aimed for a more satisfactory means to carry out the actual transformation. To conclude, I think that Frege was never content that he had succeeded in codifying the transformation from concepts to extensions or courses-of-values by way of a *primitive* law of logic, but I see no textual or systematic evidence for the much stronger claim that Frege had second thoughts concerning the legitimacy of the proposed transformation as such.

In his answer to Russell's damaging letter, dated six days after Russell sent his letter, Frege stated that:

> Your discovery of the contradiction has surprised me beyond words and, I should almost like to say, left me thunderstruck, because it has rocked the ground on which I meant to build arithmetic. (Frege 1980, p. 132)

In light of this immediate reaction to Russell, we could turn the table against Dummett and ask: "How could Russell's paradox possibly leave Frege thunderstruck, given that he already doubted that Basic Law V is true?" Our answer to this question is simply that Russell had proved something Frege had not even dreamt about, namely that Basic Law V is neither a basic law of logic, nor a derivable sentence; it is false.

The following passage from 1906 strongly suggests that the doubt Frege originally voiced about Basic Law V was that it was not clearly self-evident:

> And so "the extension of the concept *square root of 1*" is here to be regarded as a proper name, as is indeed indicated by the definite article. By permitting the transformation, you concede that such proper names have reference. But by what right does such a transformation take place, in which concepts correspond to extensions of concepts, mutual subordination to equality? An actual proof can scarcely be furnished. We will have to assume an unprovable law here. Of course it isn't as self-evident [*einleuchtend*] as one would wish for a law of logic. And if it was possible for there to be doubts previously, these doubts have been reinforced by the shock the law sustained from Russell's paradox. (Frege 1979, p. 182)

Frege's immediate reaction to Russell's paradox in the Appendix to Frege 1903 is quite illuminating. To begin with, Frege underscored that:

> I have never concealed from myself its [i.e. Basic Law V's] lack of self-evidence [*einleuchtend*] which the other possess, and which must properly be demanded of a law of logic, and in fact I pointed out this weakness in the introduction. (Frege 1903, p. 253)

Since Frege was still under the spell that arithmetic is, and must be, reducible to logic, he then tried to restore the logicist project by proposing an alternative weaker law, Basic Law V′, that could licence the transformation from concepts to their extensions or courses-of-values:

$$\vdash (\grave{\varepsilon}f(\varepsilon) = \grave{\alpha}g(\alpha)) \boldsymbol{=} (\forall x)[\neg((\neg x = \grave{\alpha}g(\alpha) \supset x = \grave{\varepsilon}f(\varepsilon))) \supset (f(x) \boldsymbol{=} g(x))].^{31}$$

Several scholars have pointed out that this formula leads to a paradox in case more than two objects exist (cf. e.g. Quine 1955, Geach 1956 and Dummett 1991a, pp. 4–6). Frege never again discussed this weakened version of Basic Law V, but soon gave up the program of logicism.[32]

[31] Cf. Frege 1903, p. 262. On the use of bold identity signs, cf. footnote 4 above.

[32] In his review, Linnebo suggested that I should relate the discussion of this essay to the big debate about the relationship between Basic Law V and the so-called "Hume's Principle". (The significance of this debate is evident on almost any page of Demopoulos 1995.) The point about the debate, as is demonstrated in detail in Boolos 1987 and Heck 1993, is that the sole role played by the introduction of extensions in Frege 1884 and by Basic Law V in Frege 1893, respectively, is to prove Hume's principle: "The number of Fs = The number of Gs if, and only if, $F \approx G$." ("The Fs and the Gs are in one-to-one correspondence" has its standard, second-order, explicit definition) (cf. Frege 1884, §63). Stripped of their technical jargon, the two sentences are given as pair C in Section 5 above, where I claimed that they express the same content or sense; thus I claim that Frege held Hume's principle to be self-evident. Boolos and Heck, Jr. have proved that if Hume's principle is added as a basic law to the informal account of Frege 1884 or to the Begriffsschrift of Frege (1893, 1903), instead of invoking extensions or courses-of-values, a consistent logic results. The puzzling feature about Hume's principle, from the perspective of this essay, is that although it is self-evident, it cannot be a basic law, but must be derived.

Heck, Jr. pinpoints a number of interpretative puzzles that arise from this peculiar relationship between Basic Law V and Hume's principle:

The questions to which we really need answers are thus: What, exactly, does Frege mean by his question, how we apprehend logical objects? [From a letter to Russell: "How do we apprehend logical objects? And I have found no other answer to it than this, We apprehend them as extensions of concepts, or more generally, as courses-of-values of functions. I have always been aware that there were difficulties with this, and your discovery of the contradiction has added to them; but what other way is there?" (Frege 1980, p. 141)] and what is the real point of the Caesar problem? In what, for Frege, does it consist that a truth is a "primitive" truth? One which "neither need[s] nor admit[s] of proof?" … And, if Frege would not have been opposed to accepting Hume's principle as a primitive truth, perhaps there was some obstacle, in his view, to accepting it as a primitive *logical* truth? … [T]he questions … need answers: For, until we have such answers, we shall not understand the significance [Basic Law V] had for Frege, since we shall not understand why he could not abandon it in favor of Hume's principle; and that is to say that we shall not understand how he conceived the logicist project. (Heck 1993, p. 287)

Obviously, a detailed treatment of these intriguing questions would demand a number of essays, and it is not clear to me how to answer them all. But, I do believe that the opening sentence of Frege 1884, §62, "How, then, are numbers to be given to us, if we cannot have any ideas (*Vorstellung*) or intuition (*Anschauung*) of them", seen against the background of the "established" results of the earlier part of Frege 1884, that numbers are objects and that "the content of a statement of number is an assertion about a concept" (§46), would constitute the framework for the analysis.

From the present perspective it might be conjectured that Frege never again returned to Basic Law V′, because it could not be claimed, on the basis of any reasonable account of the notion of content or sense, that it is self-evident. For clearly, the expressions flanking the principal identity sign do not have the same content or sense. Thus, this "law", even if, *per impossible*, true, stands in at least as much need of a proof as does "$2 + 2 = 4$". This means that even if Frege was aware of the paradox of Basic Law V′, he knew that it could not be fixed in a way that would satisfy his strong demands for an expression to be a basic law of logic. To base logicism on a law such as Basic Law V′ would be to turn the program into dogmatism, it would be devoid, by Frege's own standards, of any reasonable philosophical underpinning. Russell's version of logicism, avoiding Russell's paradox by being founded on the theory of types and the axioms of reducibility and infinity, is dogmatic in a way Frege would never have accepted.

I shall restrict myself to a rough answer to the issue that I find most pressing for the reading developed in this essay, *viz.* Frege's reasons for holding the view that Hume's principle cannot be a basic law of logic, although it is self-evident. Frege, rightly or wrongly (cf. Dummett 1991a), thought that the analysis of the transformation from "Line a is parallel to line b" to "The direction of line a is identical with the direction of line b" (Pair A in Section 5, above) applies also to Hume's principle. Although the two sentences express the same content, the singular terms "the direction of line a" and "the direction of line b" are not introduced in an appropriate way, that is, they do not fulfill the demand that "If we are to use the symbol a to signify an object, we must have a criterion for deciding in all cases whether b is the same as a, even if it is not always in our power to apply this criterion" (Frege 1884, §62). The reason is that since the contents of the two singular terms are fixed contextually, that is to say, their contents are parasitic upon the original notion of parallelism, there will be identity statements involving these singular terms whose truth-value cannot, even in principle, be decided (cf. §§65–67). "What we lack", Frege says, "is the concept of direction" (§66), i.e. a concept obtained and specified *independently* of the transformation. The same situation, Frege thought, holds with respect to the transformation from "$F \approx G$" to "The number of Fs = the number of Gs"; in this case we do not have an independent grasp on the concept of a number. I noted, in Section 5 above, that in an identity relation "$\S a = \S b$", obtained from an equivalence relation "$\Phi(a, b)$", the concept $\S\xi$, might, but need not, be known in advance; in these cases it is not known in advance.

The transformation involved in Basic Law V distinguishes itself sharply from these transformations in that the concept of a course-of-value has been introduced and specified *independently* of the equivalence relation underlying the transformation. I think this is why Frege thought that each and every identity statement containing expressions for a course-of-value could be decided. (Of course, in light of the problems often pointed out concerning Frege 1893, §10, I would not argue that Frege is right in this view; cf. e.g. Demopoulos 1995, pp. 10–12 and Dummett 1991a.) Basic Law V, then, is needed as a means for forming logical proper names that fulfil the demands given in Frege 1884, §62, as this task cannot be fulfilled by Hume's principle. That is to say, Frege took it for granted that the purely logical proper names for numbers must be given by way of explicit definitions.

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SENSE AND OBJECTIVITY IN FREGE'S LOGIC Gilead Bar-Elli The essentials of Frege's revolutionary logic appeared in his Begriffsschrift (B, 1879). Important aspects of its philosophical basis, and its significance for the foundations of mathematics, appeared in The Foundations of Mathematics (FA, 1884).Â incorporated in the mature, authoritative exposition of his logic in his magnum opus: Basic Laws of Arithmetic (BL), whose first volume appeared in 1893. What is the role of the notion of sense and of the distinction between sense and reference in Frege's logic? Is there a systematic connection between the two points (a) and (b) mentioned above, so that their being incorporated together in Frege's mature logic is not accidental?