Workshop

WHEN HPM MEETS MKT – EXPLORING THE PLACE OF HISTORY OF MATHEMATICS IN THE MATHEMATICAL KNOWLEDGE FOR TEACHING

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In mathematics education, one way to approach the question of "What should a mathematics teacher know?" is through the framework of Mathematical Knowledge for Teaching (MKT), based on the work of Deborah Ball and others. In most articles on MKT, history of mathematics is barely mentioned or not mentioned at all. However, there are exceptions pointing out that history of mathematics has a role in several – or indeed all – the subdomains of the MKT model. A further exploration of this can give important insights into the discussion on the role of history of mathematics in teacher education and in mathematics teaching, as well as enhance the MKT theory with insights from the work on history of mathematics.

In this article, I discuss didactical examples from the literature related to The International Study Group on the relations between the History and Pedagogy of Mathematics (HPM), to illustrate and discuss the role of history of mathematics in the framework of MKT.

INTRODUCTION

A fundamental question for mathematics teacher educators is "What should a mathematics teacher be able to do?" An important sub-question is "What should a mathematics teacher know?" Of course, the mathematics teacher needs to know everything that the students are supposed to learn, and more than that. What is this "more than that"? Is it more advanced mathematics, is it mathematical games, is it the history of mathematics, or is it simply everything the teacher educator finds interesting?

One way to approach the question of "What should a mathematics teacher know?" is through the framework of Mathematical Knowledge for Teaching (MKT), based on the work of Deborah Ball and others (Ball, Thames, & Phelps, 2008). This framework (often referred to as "the oval") is presently very popular in mathematics education research, and the article I just cited is referred to in hundreds of scholarly articles per year. The purpose of the present article (and of the workshop on which it was based) is to discuss whether MKT is useful for the discussions of what history of mathematics may contribute to in mathematics teaching; but also, how can history of mathematics contribute to the development of the MKT framework? The discussions are based on investigation of particular examples of the use of history in mathematics teaching. In the workshop, the group discussions were briefly summarized to everyone, and these summaries have informed the discussions in this article. First, I will give a short introduction to the MKT framework.

INTRODUCTION TO MKT

In developing their framework, Deborah Ball and others have taken Shulman (1986) as their starting point, with his concepts of "subject matter knowledge" and "pedagogical content knowledge". In the MKT framework, these two domains have been subdivided further, based on research on practice in the US.

I will introduce each of these domains shortly, but I would like to stress at the beginning that I see the mathematical knowledge for teaching as dependent on the context. What fits into which domain depends on the teacher, on the students, on the curriculum etc. In my examples, I will mostly think of a Norwegian teacher who is teaching grade 6 or 7 (students aged 12-14) or something similar. A teacher teaching other students, in another grade level, at another point in time or in another country, may need different knowledge. When discussing MKT, it is important to be explicit about which teacher and context we are thinking of. It can also be argued that the contents of the domains move around at times of curricular change. (Smestad, Jankvist & Clark, 2014)

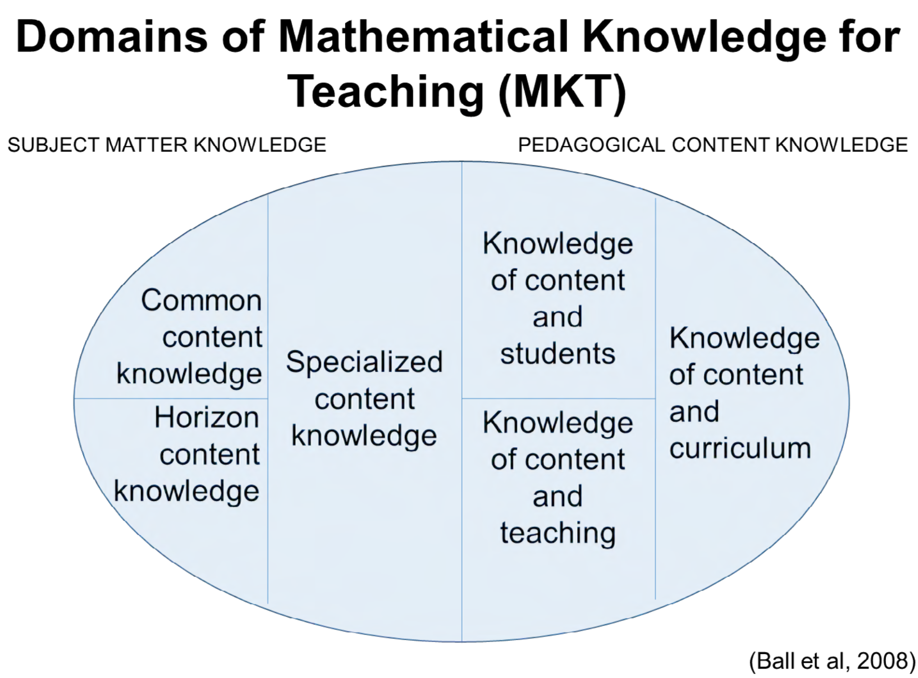

Figure 1: Domains of Mathematical Knowledge for Teaching, from Ball et al. (2008).

On the left-hand side of the diagram (Figure 1), we have subject matter knowledge:

- *Common content knowledge* is knowledge in mathematics that is not special for teachers. This includes the mathematics the students are supposed to learn, such as being able to add two integer numbers.
- *Specialized content knowledge* is knowledge in mathematics that is primarily necessary for teachers. Ball et al. (2008, p. 404) observed: “it is hard to think of others who use this knowledge in their day-to-day work”. Ball et al. mentioned as an example the ability to see a new algorithm and decide whether it is sound, which is something a teacher needs when evaluating students’ attempted solutions. The importance of this domain is to point out that there are things teachers need to know about mathematics that other professionals using mathematics do not need.
- *Horizon content knowledge* can be described as “a sense for how the content being taught is situated in and connected to the broader disciplinary territory” (Jakobsen, Thames, Ribeiro, & Delaney, 2012, p. 4642) This includes how the content taught is connected to other mathematical topics the students will meet later, but also for instance how the mathematical content has developed.

On the right-hand side, we have pedagogical content knowledge:

- *Knowledge of content and students* is “focused on how *students* think about, know and learn mathematics” (Mosvold, Jakobsen, & Jankvist, 2014, p. 50, my emphasis), for instance, identifying student misconceptions.
- *Knowledge of content and teaching* concerns “design of instruction” (Mosvold, et al., 2014, p. 50) in mathematics, for instance how to design a lesson with the use of examples, tasks and discussions.
- *Knowledge of content and curriculum* is “a particular grasp of the materials and programs that serve as ‘tools of the trade’ for teachers”. (Shulman, 1987, p. 8, cited in Mosvold et al., 2014, p. 50). This includes both knowledge of the curriculum documents – which mathematical topics have the students met before and which will they meet next year – and which resources are available.

I also stress that all mathematical knowledge for teaching does not fit nicely into these categories. Often, there will be knowledge that fits into more than one. The point of the model, the way I see it, is not to make everything nice and orderly, but to give us additional tools to use while discussing.

Next, I give some examples of mathematical knowledge for teaching, and discuss what would be the “right” domains to place them in. (In the workshop, this was discussed in groups.)

Example 1: Knowing how to calculate 325:25

Of course, no teacher should attempt to teach division knowing only this. Thus, it is tempting to try to list (or map) other bits of knowledge we would like mathematics teachers to have for teaching division, including on its history, for them to be able to teach it in a meaningful way. Indeed, the MKT domains may serve as starting points for such a list, and this could form a meaningful activity in teacher education. However, the present task was to look at just this isolated piece of knowledge. In that case, this would clearly be common content knowledge, as everybody in our society is supposed to learn how to divide (disregarding arguments that this is a task better done by computers or calculators).

Example 2: Knowing how to simplify $(4x^3+2x^2-x) : x$

This is an example of subject matter knowledge. As everyone is supposed to learn this at some point, it cannot be said to be special for teachers, so it is common content knowledge. However, as it is not something the teacher's students are supposed to learn at this age (12–14) in Norway, but is connected to what they are supposed to learn at a later stage, it could be argued that it should rather be placed in the domain of horizon content knowledge for this teacher. The knowledge that the students will have to learn this later, however, would be knowledge of content and curriculum for the teacher.

Example 3: Knowing a little about the historical origin of the Hindu-Arabic numeral system.

I argue that this is common content knowledge. Knowing a little about our numeral system should be part of the mathematics curriculum for everyone. In the 1997 curriculum of Norway, it was explicitly so, and even in the new curriculum, there are enough general remarks on the importance of history that it should be included.

This is a good opportunity to discuss what "mathematics" is in the MKT context. In the original articles by Ball and her colleagues, I cannot find "mathematics" or "mathematical" defined. Examples of common content knowledge are, for instance, "a simple subtraction computation" (Ball, et al., 2008, p. 396), and common content knowledge is defined in terms of computations, "simply calculating an answer or, more generally, correctly solving mathematics problems." (Ball, et al., 2008, p. 399). However, when discussing what a mathematics teacher should know, it would not make sense to define "mathematics" narrowly as only mathematical algorithms or mathematical concepts. Arguably, at least in some parts of the world the history of mathematics is an intrinsic part of the subject mathematics (Fauvel & van Maanen, 1997), and in so far as history of mathematics is part of what students are supposed to learn in mathematics class, I would regard this as subject matter knowledge, more precisely common content knowledge.

Here, we can also touch upon Jankvist's concepts of history of mathematics as tool vs. goal (Jankvist, 2009). If we see history of mathematics as a goal, meaning that students should learn something from history of mathematics that they cannot learn otherwise, it is reasonable to think of the history of mathematics in question as part of common content knowledge. If it is just a "tool" for learning mathematics (in the sense of algorithms and concepts), it becomes purely a pedagogical device, perhaps better placed in knowledge of content and teaching or knowledge of content and curriculum.

Thus, we see that decisions on where to place parts of teacher knowledge in the various MKT domains depend on our context, including our goals for teaching mathematics, which in its turn are connected to our view of what "mathematics" is.

Example 4: Knowing how to find two fractions with different (and small) denominators, that adds up to a number less than 1.

This knowledge is certainly useful while teaching fractions. I argue that not many others need exactly this knowledge. Therefore, it is specialized content knowledge.

Example 5: Knowing that the Egyptians mostly used unit fractions (that is, fractions with numerator 1).

Keeping to my point of view that history of mathematics is part of mathematics, I would regard this as knowledge of mathematics that is useful for the teacher, as it may influence him when introducing fractions to the students. Thus, it is either horizon content knowledge or specialized content knowledge – and it could be considered both.

Example 6: Finding the mistake in a calculation and considering whether it could be a sign of a usual misconception.

Being able to find a mistake in a calculation (as opposed to just finding that the answer is wrong) is mathematical knowledge that you rarely need outside of the teaching profession. Thus, it should be considered as specialized content knowledge. Knowing about typical misconceptions, on the other hand, is knowledge of content and students. In this way, threads of knowledge that are closely knit together and used simultaneously when planning or performing teaching, may well belong to different domains of the MKT model.

Example 7: Finding a useful counterexample to the sentence "Division of a number by another number makes the original number smaller."

Finding a counterexample is specialized content knowledge as other professions rarely need counterexamples, while they are necessary for teachers. However, the word "useful" is important. A teacher cannot use just any counterexample; he needs the counterexample that is exactly right for his students. To find that, he also needs to know his students well, so knowledge of content and students is needed. As in example 6, we see how the knowledge from different domains work together even in simple tasks.

Example 8: Being able to figure out whether the method of multiplication in Figure 2 works in general. (From Smestad, 2002)

26·41	
13·82	82
12·82	
6·164	
3·328	328
2·328	
1·656	656
	1066

Figure 2: Method of multiplication

Being able to figure out why or whether methods work, is not normally part of the mathematics that everybody learns and needs, but it is central knowledge for teachers, who need this to be able to give relevant feedback to students using different methods. Thus, this is specialized content knowledge. In some countries, this method (which is inspired by the Egyptian method for multiplication given in the Ahmes papyrus) could be part of the curriculum, and would then be common content knowledge.

Example 9: Knowing the origin of the words "algebra" and "algorithm".

Again, I consider history of mathematics as an intrinsic part of mathematics. In most countries and curricula, the etymology of words will not be considered part of what everybody needs to learn, thus it is probably not common content knowledge. However, to be able to answer reasonable questions from students, a teacher needs to know this part of mathematics. Therefore, it should be considered specialized content knowledge. Since it is part of the background of the mathematics teachers teach, it could also be argued that it is horizon content knowledge. However, if you do not regard history of mathematics as part of mathematics, this knowledge would be relegated to the pedagogical content knowledge domains, and would perhaps belong in knowledge of content and teaching.

The first nine examples were decontextualized sentences about knowledge. Some participants in the workshop found it very frustrating to work on such examples, stressing that the context is so important. The remaining examples are (parts of) teaching materials concerning history of mathematics that are designed for teacher education. Here, I discuss what knowledge could result from working on the examples and in which domain(s) such knowledge belongs. In the text, I will refer to the preservice teachers as PSTs, while their future students in school will be referred to simply as "students".

Example 10: Biographical introductions

Below is an example from Haanæs & Dahle (1997, p. 97), a textbook for 6th grade. (It was mentioned in Smestad (2002, p. 36). The translation is mine.):

> We consider Florence Nightingale the founder of modern nursing, but she was also one of the first female statisticians in the world. She considered statistics a way of changing society, and she contributed to making statistics a subject on its own at the university of Oxford in England. She tried to help people who were ill and suffering in the world by showing how many they were. She made statistics herself that lead to a new way of treating patients all over the world.

Imagine that we work with PSTs on preparing such introductions: what knowledge could result, and which domain would the knowledge belong to?

Writing such biographical texts should result in some knowledge of the biography of a mathematician. Is this just "spice" that the PSTs can later use to engage their students? In that case, it is purely a pedagogical tool, probably belonging to knowledge of content and curriculum. However, one of the PSTs' goals may be to develop in the students a sense that mathematics is a human activity and to develop students' epistemological points of view. In that case, the PST needs biographical information "on the horizon" of the mathematics to do that, thus the biographical details are horizon content knowledge. Some PSTs may even think that knowledge of a particular mathematician's life is part of what everyone should know (for instance Abel in Norway or Newton in England) – just like some teachers think everyone should know about Ibsen (in Norway) or Shakespeare (in England). In that case, they will consider it common content knowledge.

Again, we see that knowledge cannot be placed in MKT domains without regarding the personal epistemology and the goals of the teacher.

Example 11: al-Khwarizmi

The following is an excerpt from Clark (2012, pp. 72-73), where she described part of a Using History for Teaching course. In this task prospective mathematics teachers (PMTs) are

> …solving quadratic equations using the methods of al-Khwarizmi (early ninth century CE). PMTs in the UsingHistory course were provided an excerpt from the History of Mathematics: A Reader (Fauvel & Gray, 1987, pp. 228-231) in which an English translation of al-Khwarizmi's rhetorical and geometric explanation for how to solve quadratic equations was given. A similar excerpt is:
>
> > … a square and 10 roots are equal to 39 units. The question therefore in this type of equation is about as follows: what is the square which combined with ten of its roots

> will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this, which is 8, subtract from it half the roots, 5 leaving 3. The number three therefore represents one root of this square, which itself, of course is 9. Nine therefore gives the square. (O'Connor & Robertson, 1999, para. 12)

> […] PMTs were not given the accompanying figure at the outset of the exploration. Instead, they were to use the rhetorical solution to develop the geometric argument. After exploring each of al-Khwarizmi’s explanations and reporting out the whole class, groups continued with their investigation of solving quadratic equations from a historical perspective […].

What knowledge could such work lead to, and which domain(s) would it fit in?

In Clark (2012), an important point was that the PSTs learned how to solve equations geometrically. That knowledge is specialized content knowledge, if not common content knowledge. However, we hope that PSTs will not only remember the method of solving, but also realize something about the long history of mathematics and the point that different cultures have contributed to its development. The cultural component is horizon content knowledge.

Example 12: Fibonacci

Here is the start of the introduction of Fibonacci’s *Liber abaci* (1202):

> After my father's appointment by his homeland as state official in the customs house of Bugia for the Pisan merchants who thronged to it, he took charge; and in view of its future usefulness and convenience, had me in my boyhood come to him and there wanted me to devote myself to and be instructed in the study of calculation for some days.
>
> There, following my introduction, as a consequence of marvelous instruction in the art, to the nine digits of the Hindus, the knowledge of the art very much appealed to me before all others, and for it I realized that all its aspects were studied in Egypt, Syria, Greece, Sicily, and Provence, with their varying methods; and at these places thereafter, while on business.
>
> I pursued my study in depth and learned the give-and-take of disputation. But all this even, and the algorism, as well as the art of Pythagoras, I considered as almost a mistake in respect to the method of the Hindus. (Modus Indorum). Therefore, embracing more stringently that method of the Hindus, and taking stricter pains in its study, while adding certain things from my own understanding and inserting also certain things from the niceties of Euclid's geometric art, I have striven to compose this book in its entirety as understandably as I could, dividing it into fifteen chapters.
>
> Almost everything which I have introduced I have displayed with exact proof, in order that those further seeking this knowledge, with its pre-eminent method, might be instructed, and further, in order that the Latin people might not be discovered to be

> without it, as they have been up to now. If I have perchance omitted anything more or less proper or necessary, I beg indulgence, since there is no one who is blameless and utterly provident in all things.
>
> The nine Indian figures are:
>
> 9 8 7 6 5 4 3 2 1
>
> With these nine figures, and with the sign 0 ... any number may be written.

What knowledge could work on the introduction of *Liber abaci* contribute to, and into which domain(s) does this knowledge belong?

I have used this example with my PSTs, and I had two main goals: The first was that PSTs should have some idea of where the Hindu-Arabic numeral system comes from. I would regard this as horizon content knowledge. (At least as long as the curriculum does not mandate that everyone should learn about this, in which case it would be common content knowledge). The second was more general: that the PSTs should realize that the mathematics we use is not predetermined but evolves, depending on human choices. This would also be horizon content knowledge.

Example 13: The Pascal-Fermat correspondence

The Pascal-Fermat correspondence, which many regard as the beginning of probability theory, provides interesting questions. I have developed a few exercises based on the history, which can be found in Smestad (2012). They lead PSTs through some of the problems found in the correspondence. In the workshop at the Seventh European Summer University, we looked at what is known as the problem of points, and the PSTs' work on different mathematicians' attempts at solutions.

Again, I have used these exercises in my own teaching. My main goal in using them was that the PSTs should learn probability theory while doing them. This would be common content knowledge. But I also included these exercises because they showed how mathematics has developed, and in particular how mathematicians interact, using trial and error and providing counterexamples to develop theories. This I would regard as horizon content knowledge. But again, the exercises could also be seen as mainly motivational, with a little "human interest" added to spice up the mathematics. In that case, I would regard it as knowledge of content and curriculum as "tools of the trade" to teach probability.

Concluding discussion

The examples show that what fits in which domains is context-dependent. This would be a bad thing if the point was to sort knowledge neatly into domains (or to test PSTs). For use in teacher education to foster discussion, it may not be. In discussing examples in connection to MKT, PSTs need to be explicit about the way they intend the history to be used and in which context they intend to use it. It becomes clear that the same knowledge can be used for different goals, depending on the teachers' personal epistemology and his goals for teaching mathematics.

In the workshop, we also discussed what could come out of connecting HPM and MKT. Already, several articles have been written about how looking at history of mathematics can enrich the MKT framework. (Mosvold, et al., 2014; Smestad, et al., 2014) In this article, I argued that researchers should be explicit about what they regard as "mathematics" or "mathematical" when looking at "mathematical knowledge for teaching" to include history of mathematics. Also, from a more strategic point of view, the HPM community should engage with the theories that are considered important in the general mathematics education community, as seen, for instance, in the PME conferences. If we do not, we risk being seen as irrelevant, which will make it more difficult to get our ideas across and to attract newcomers. In addition, in teacher education it is useful to connect to theories that PSTs already know when discussing history of mathematics.

The advantages of using MKT in HPM research are less clear. It is important that we are explicit about how history of mathematics may contribute to mathematics teaching, but there are other frameworks for this that may be as useful as the MKT framework, for instance Jankvist's idea of history as a tool vs. history as a goal.

We should not just look at what can be gained by using the MKT framework, but also at what can be lost. In his 2014 book *The Beautiful Risk of Education*, Gert Biesta argued forcefully that too much ink is spent on the qualification of teachers, and not enough on the socialization and – most importantly in his view – the "subjectification" of teachers. "Subjectification" is connected to his concept "becoming educationally wise", which he argued is very different from obtaining knowledge. (Biesta, 2014) Instead of looking at what teachers should know (which the MKT framework helps us do), we should perhaps look at how teachers can use what they know in the art of teaching. As with any frameworks of interest to mathematics teacher education, the discussions on what it does not encompass are as important as the framework itself.

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Welcome to the story of mathematics. What is mathematics? Mathematics may be defined as â€œthe study of relationships among quantities, magnitudes and properties, and also of the logical operations by which unknown quantities, magnitudes, and properties may be deducedâ€ (according to Microsoft Encarta Encyclopedia) or "the study of quantity, structure, space and change" (Wikipedia).Â My intention is to introduce some of the major thinkers and some of the most important advances in mathematics, without getting too technical or getting bogged down in too much detail, either biographical or computational. Teaching mathematics should give students the opportunity to â€œdo mathematics.â€ In other words, although the â€œpolished productsâ€ of mathematics form the part of mathematical knowledge that is communicated, criticized (in order to be finally accepted or rejected), and serve as the basis for new work, the process of producing mathematical knowledge is equally important, especially from a didactical point of view. Perceiving mathematics both as a logically structured collection of intellectual products and as processes of knowledge production should be the core of the teaching of mathematics. At t