INFINITESIMAL SYSTOLIC RIGIDITY OF METRICS ALL OF WHOSE GEODESICS ARE CLOSED AND OF THE SAME LENGTH

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Abstract. We show that Finsler manifolds all of whose geodesics are closed and of the same length satisfy an infinitesimal isosystolic inequality to all orders.

"Mon intérêt pour le problème dont je veux vous entretenir ici, je le dois à un ami ébéniste."
Arthur Besse.

1. Introduction

In this paper we're interested in studying the infinitesimal aspects of the relationship between the volume of a closed Riemannian or Finsler manifold and the length of its shortest closed geodesic. More precisely, we define the *systolic volume* of a closed n-dimensional Finsler manifold as the quotient

$$\mathfrak{S}(M,F) = \frac{\mathrm{vol}(M,F)}{\mathrm{sys}(M,F)^n}\,,$$

where vol denotes the (Holmes-Thompson) volume and sys denotes the *systole*—the length of a shortest closed geodesic, consider $\mathfrak{S}$ as a function on the space of Finsler metrics on M, and propose to study its infinitesimal behaviour.

The starting point of our investigation was the question of whether the standard metrics on compact rank-one symmetric spaces are local minima of the systolic volume. One would at least like to know whether they are critical points in some sense. In this note we shall prove this is indeed the case not only for compact rank-one symmetric spaces, but also for the more general class of (reversible and non-reversible) Finsler manifolds all of whose geodesics are closed and of the same length.

The programme of studying the local and infinitesimal behaviour of the systolic volume was pioneered by Berger [7] in the case of Riemannian metrics on real projective spaces and by Balacheff ([5], [6]) in the case of Riemannian metrics on the two-sphere. In these works we find two possible ways

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around the difficulty posed by the lack of differentiability of the systolic volume (even worse, the function is not everywhere continuous, although by the results of Weinstein [21] and Bottkol [9] it is continuous at Finsler metrics all of whose geodesics are closed and of the same length). In this paper we follow the approach of Balacheff illustrated by the following result ([5]): *if g_t is a differentiable deformation by smooth riemannian metrics of the standard metric g_0 of S^2, then there exists another Riemannian deformation h_t which coincides with g_t to first order (i.e., $g_t = h_t + o(t)$) such that*

$$\mathfrak{S}(S^2, h_t) \geq \mathfrak{S}(S^2, g_0).$$

It is in this sense that we consider (S^2, g_0) as a critical point of the systolic volume.

Let us say that (M, g) is *infinitesimally rigid to order N* if for any differentiable Finsler deformation g_t of the metric g, there exists another Finsler deformation h_t which coincides with g_t to order N (i.e., $g_t = h_t + o(t^N)$) and such that

$$\mathfrak{S}(M, h_t) \geq \mathfrak{S}(M, g).$$

We will establish the following result:

Theorem 1. *A closed Finsler manifold all of whose geodesics are closed and of the same length is infinitesimally rigid to all orders.*

In particular, the following metrics are infinitesimally rigid to all orders:

- The standard metrics on compact rank-one symmetric spaces (e.g., real, complex, and quaternionic projective spaces, spheres, and the Cayley plane).
- Zoll metrics on the two-sphere (see [8] and [12] for a wealth of constructions).
- Weinstein's higher-dimensional analogues of Zoll metrics of revolution on higher-dimensional spheres ([8]).
- Projective Finsler metrics on spheres and real projective spaces (see [16], [19], and [2] for constructions of these metrics)..

A remark on the limitations of our proof of Theorem 1 is that if we start with a Riemannian deformation g_t, the approximating deformation h_t for which the isosystolic inequality holds may not be Riemannian, so that the present paper does not quite supersede [5]. Likewise, if we start with a deformation of reversible Finsler metrics, the approximating deformation could, in principle, consist of non-reversible metrics. Lest these remarks dampen the reader's enthusiasm, we mention that there are very many (reversible and non-reversible) Finsler deformations to a given metric and the fact that compact rank-one symmetric spaces are infinitesimally rigid in our sense is much stronger than if we restrict the deformations to be Riemannian. The short survey on systolic Finsler geometry in Section 5 will help the reader get a feel for how the introduction of Finsler metrics—and their various notions of volume—changes (or not) the systolic landscape.

2. Hamiltonian formalism for isosystolic inequalities

In what follows we will not work with Riemannian or Finsler metrics, but with the more general and simpler notion of *star Hamiltonian:*

Definition 2.1. Let M be a smooth closed manifold. A (smooth) *star Hamiltonian* is a continuous function $H : T^*M \to [0,\infty)$ that is

- positive and smooth outside the zero section;
- positively homogeneous of degree one (i.e., $H(\lambda p) = \lambda H(p)$ whenever $\lambda > 0$);
- proper.

Since a star Hamiltonian H is proper, the hypersurface $\{H = 1\}$ is compact. By analogy with Riemannian geometry, we shall call it the unit co-sphere bundle of H and denote it by $S^*_H M$. Roughly speaking, just as a Riemannian metric can be seen as a choice of an ellipsoid in every tangent or cotangent space, a star Hamiltonian is a choice of a star-shaped body in every cotangent space.

The ingredients of isosystolic inequalities are volume and the lengths of closed geodesics. The canonical one-form α on the cotangent bundle allows us to generalize these objects to star Hamiltonians. More precisely, instead of closed geodesics and their lengths we shall consider closed characteristics and their actions; instead of the volume of a Riemannian manifold we shall consider the Liouville volume of the unit co-sphere bundle.

Definition 2.2. Let H be a star Hamiltonian defined on the cotangent bundle of an n-dimensional compact manifold M and let $S^*_H M$ be its unit co-sphere bundle. A closed differentiable curve γ on $S^*_H M$ is a *closed characteristic* if it is an integral curve of the Hamiltonian vector field X_H defined by the equation $d\alpha(X_H, \cdot) = -dH(\cdot)$. The *action* of γ is defined as the integral of α over γ and the *Liouville volume* of $S^*_H M$ is given by

$$\mathrm{vol}(S^*_H M) = \frac{1}{n!}\int_{S^*_H M} \alpha \wedge (d\alpha)^{n-1}.$$

The solution of the Weinstein conjecture on cotangent bundles by Hofer and Viterbo (see [13]) implies that there is always at least one closed characteristic on the co-sphere bundle of a star Hamiltonian. By analogy with the Riemannian case, we define the systole as the least action of a closed characteristic:

$$\mathrm{sys}(S^*_H M) = \inf\{\int_\gamma \alpha \mid \gamma \text{ closed characteristic}\}$$

and the systolic volume of $S^*_H M$ as

$$\mathfrak{S}(S^*_H M) := \frac{\mathrm{vol}(S^*_H M)}{\epsilon_n \mathrm{sys}(S^*_H M)^n},$$

where ϵ_n is the volume of the Euclidean unit ball of dimension n.

The main result can now be stated as follows:

Theorem 2.1. *Let M be a smooth closed manifold and consider a smooth family of star Hamiltonians H_t on T^*M such that the Hamiltonian flow of H_0 is periodic of period* $\mathrm{sys}(S^*_{H_0}M)$. *Then for any positive integer N there exists another deformation K_t of H_0 by star Hamiltonians with $K_t = H_t + o(t^N)$ and such that*

$$\mathfrak{S}(S^*_{K_t}M) \geq \mathfrak{S}(S^*_{H_0}M).$$

We remark that the real importance of the generalized setup is to reveal the group of homogeneous symplectic transformations as the group of symmetries in systolic geometry.

Definition 2.3. Let $T^*M \setminus 0$ denote the cotangent bundle of M with the zero section removed. A diffeomorphism

$$\psi : T^*M \setminus 0 \longrightarrow T^*M \setminus 0$$

is a *homogeneous symplectic transformation* if it satisfies one of the following equivalent conditions:

- $\psi^*\alpha = \alpha$;
- $\psi^* d\alpha = d\alpha$ and $\psi(\lambda\xi) = \lambda\psi(\xi)$ for $\lambda > 0$.

Notice that if H is a star Hamiltonian and ψ is a homogeneous symplectic transformation, then $H \circ \psi$ is a star Hamiltonian and its Hamiltonian flow is conjugate to that of H. Moreover, ψ preserves the action of characteristics and the volume of the unit co-sphere bundles. In particular, $\mathfrak{S}(S^*_{H\circ\psi}M) = \mathfrak{S}(S^*_H M)$.

The group of homogeneous symplectic transformations on $T^*M \setminus 0$ is a natural generalization of the group of diffeomorphisms of M : if $f : M \to M$ is a diffeomorphism, its lift f^* to T^*M is a homogeneous symplectic transformation. However, as the next (folklore) construction shows, there are many more examples.

Proposition 2.1. *Let M be a closed manifold and let $H : T^*M \setminus 0 \to \mathbb{R}$ be a smooth function that is positively homogeneous of degree one. There exists a positive time τ such that for every point $p \in T^*M \setminus 0$ the integral curve $\psi_t(p)$ of the Hamiltonian vector field X_H with initial condition p is defined in the interval $(-\tau, \tau)$ and for every t in this interval*

$$\psi_t : T^*M \setminus 0 \longrightarrow T^*M \setminus 0$$

is a homogeneous symplectic transformation.

Proof. For $\lambda > 0$, let us denote by δ_λ the dilation map $\delta_\lambda(p) = \lambda p$ defined on the cotangent bundle of M. By the homogeneity of H and of the symplectic form $\omega = -d\alpha$, the Hamiltonian vector field of H satisfies

$$D\delta_\lambda(X_H(p)) = X_H(\lambda p).$$

It follows that if $\psi_t(p)$ is the integral curve of X_H starting at p, $\delta_\lambda(\psi_t(p))$ is (part of) the integral curve starting at λp. This already allows us to conclude that if for a given time t the map ψ_t is defined on the whole of $T^*M \setminus 0$, then it is a homogeneous symplectic transformation.

Since the unit co-sphere bundle $\Sigma \subset T^*M$ of an (auxiliary) Riemannian metric on M is compact, there is a positive time τ such that for every point $p \in \Sigma$ the integral curve $\psi_t(p)$ is defined in the interval $(-\tau, \tau)$. By homogeneity, we see that ψ_t is defined on the whole of $T^*M \setminus 0$ for all t in this interval. □

As a final remark on homogeneous symplectic transformations, we mention that if H is the Hamiltonian of a Finsler metric and ψ is a homogeneous symplectic transformation sufficiently close to the identity, then $H \circ \psi$ is again the Hamiltonian of a Finsler metric. However, if the original metric was Riemannian or reversible, the new metric is not necessarily so.

We close this section by explaining why Theorem 2.1 is a generalization of Theorem 1. Recall the definition of a (not-necessarily reversible) Finsler metric on M: a function

$$F : TM \longrightarrow [0, \infty)$$

that is continuous, smooth outside the zero section, positively homogeneous of degree one (i.e. $F(tv) = tF(v)$, $t > 0$), and such that for every nonzero tangent vector v, the quadratic form

$$g(v) := \frac{1}{2}\frac{\partial^2 F^2}{\partial v_i \partial v_j}(v)$$

is positive definite.

The *Legendre transform associated to* F is the map $\mathcal{L} : TM \setminus 0 \to T^*M \setminus 0$ defined by $v \mapsto g(v)(v, \cdot)$. This map is a diffeomorphism which sends fibers to fibers and is homogeneous of degree one.

To the Finsler metric F we associate the star Hamiltonian $H = F \circ \mathcal{L}^{-1}$. It is well-known (see sections 3.5–3.7 of [1]) that characteristics on $S^*_H M$ project down to geodesics on M and that, conversely, if σ is a geodesic on M parameterized by arclength, the curve $\mathcal{L} \circ \dot\sigma$ is a characteristic on $S^*_H M$. The length of the geodesic equals the action of the corresponding characteristic.

By definition, the volume $\mathrm{vol}(S^*_H M)$ is equal to the Holmes-Thompson volume of (M^n, F) times the volume of the Euclidean unit ball of dimension n. For Riemannian metrics the Holmes-Thompson volume is the standard volume and, therefore, in both the Riemannian and Finsler case we have

$$\mathfrak{S}(S^*_H M) = \mathfrak{S}(M, F).$$

3. Systolic inequality for commuting Hamiltonians

The first step in the proof of Theorem 2.1 is to single out a class of perturbations for which the systolic volume increases.

Theorem 3.1. *Let H_0 and K be two star Hamitonians defined on the cotangent bundle of a closed manifold M. Assume that the Hamiltonian flow of H_0 is periodic of period $\mathrm{sys}(S^*_{H_0} M)$ and that $\mathrm{vol}(S^*_{H_0} M) = \mathrm{vol}(S^*_K M)$. If K is constant along the orbits of the Hamiltonian flow of H_0, then*

$$\mathrm{sys}(S^*_K M) \leq \mathrm{sys}(S^*_{H_0} M).$$

Moreover, equality holds if and only if $K = H_0$.

Proof. We first simplify things by working only on $S^*_{H_0} M$ at the cost of using the radial map to pull-back all important objects from $S^*_K M$. Let $\rho : S^*_{H_0} M \to (0, \infty)$ denote the restriction of the function $1/K$ to the unit co-sphere bundle of H_0 and let $\delta : S^*_{H_0} M \to S^*_K M$ be the map $\xi \mapsto \rho(\xi)\xi$. If we abuse notation and continue to denote by α the restrictions of the canonical one-form to the co-sphere bundles of H_0 and K, we can write $\delta^*\alpha = \rho\alpha$. From this we see that

$$\mathrm{vol}(S^*_K M) = \frac{1}{n!} \int_{S^*_{H_0} M} \rho^n \alpha \wedge (d\alpha)^{n-1},$$

where n is the dimension of M. Since $\mathrm{vol}(S^*_{H_0} M) = \mathrm{vol}(S^*_K M)$, the average of ρ^n over $S^*_{H_0} M$ is equal to one and therefore the minimum of ρ is less than or equal to one.

We now proceed to construct a closed characteristic on $S^*_K M$ whose action is no greater than $\mathrm{sys}(S^*_{H_0} M)$. Let ξ be a point in $S^*_{H_0} M$ where ρ attains its minimum value and let γ be the closed characteristic that passes through ξ. We claim that the image of γ under the radial map δ is a closed characteristic on $S^*_K M$ and that its action is

$$\min(\rho)\, \mathrm{sys}(S^*_{H_0} M) \leq \mathrm{sys}(S^*_{H_0} M).$$

Indeed, to see that the curve $\delta \circ \gamma$ is a characteristic in $S^*_K M$ we must verify that its velocity vectors are in the nullity of $d\alpha$. This is the same as verifying that the velocity vectors of γ are in the nullity of $\delta^* d\alpha = \rho d\alpha + d\rho \wedge \alpha$, and this follows immediately from the fact that γ is a characteristic in $S^*_{H_0} M$ and that the points of γ are critical points for the radial function ρ. As for the action of $\delta \circ \gamma$:

$$\int_{\delta\circ\gamma} \alpha = \int_\gamma \delta^*\alpha = \int_\gamma \rho\alpha = \min(\rho) \int_\gamma \alpha.$$

□

We end this section by mentioning two noteworthy applications of Theorem 3.1. The reader who wishes to pursue the proof of the main result may safely skip them.

Katok deformations and systolic inequalities. Let us recall that the *Katok deformation* of a Finsler metric admitting a non-zero Killing vector field X is the family of Finsler metrics whose Hamiltonians are given by

$$H_\epsilon(\xi_x) = H_0(\xi_x) + \epsilon \xi_x(X(x)) \quad (0 \leq \epsilon \ll 1),$$

where H_0 is the Hamiltonian of the original metric. A key property of Katok deformations is that the Hamiltonians commute with the unperturbed Hamiltonian H_0 (see [15] and [22]). The following result follows immediately from Theorem 3.1.

Corollary 3.1. *If (M, F_ϵ) is a Katok deformation of a Finsler manifold (M, F) all of whose geodesics are closed and of the same length, then*

$$\mathfrak{S}(M, F_\epsilon) \geq \mathfrak{S}(M, F)$$

with equality if and only if $\epsilon = 0$.

Application to the dynamics of rigid bodies. The motion of a rigid body with a fixed point under the influence of a conservative force is modeled by a Lagrangian of the form $L = T - V$, where the kinetic energy T is a left-invariant Riemannian metric on the rotation group $SO(3)$ and the potential energy V is a smooth function on $SO(3)$. The left-invariant metric is given by the inertia tensor of the body and the metric is bi-invariant if and only if the ellipsoid of inertia is a sphere. For a fixed energy E that is greater than the maximum of V, we can describe the motion of the body as geodesics of the *Jacobi metric* $(E - V)T$ (see Theorem 3.7.7 in the book [1] which we also recommend as a reference for all we shall use in this section on the mechanics of rigid bodies and Lie groups).

Corollary 3.2. *Let $(E-V)T$ be the Jacobi metric associated to the dynamics of a rigid body with a fixed point under the influence of a conservative force with potential energy V. We have that*

$$\mathfrak{S}(SO(3), (E - V)T) \geq \pi$$

with equality if and only if the mechanical problem is that of a free rigid body whose ellipsoid of inertia is a sphere.

Proof. By a classic theorem of Pu [17], if (M, g) is a compact, non simply-connected homogeneous space and ν is a positive smooth function on M such that the volumes of the conformal metrics (M, g) and $(M, \nu g)$ are the same, then the *non-contractible systole* of $(M, \nu g)$—the length of a shortest closed non-contractible geodesic—is smaller than that of (M, g). Equality holds if and only if ν is a constant.

If we remark that the topological systole of $(SO(3), T)$ coincides with its systole for any left-invariant metric T, then Pu's theorem implies that

$$\mathfrak{S}(SO(3), (E - V)T) \geq \mathfrak{S}(SO(3), T)$$

with equality if and only if the potential energy V is constant.

Since the Hamiltonian of any left-invariant Riemannian metric on $SO(3)$ is constant along the orbits of the geodesic flow of a bi-invariant metric T_0, Theorem 3.1 allows us to conclude that

$$\mathfrak{S}(SO(3), (E - V)T) \geq \mathfrak{S}(SO(3), T) \geq \mathfrak{S}(SO(3), T_0) = \pi$$

with equality if and only if T is bi-invariant (the ellipsoid of inertia is a sphere) and V is constant (no forces act on the body).

□

4. Normal forms and proof of Theorem 2.1

Another way of saying that a Hamiltonian K is constant along the orbits of the Hamiltonian flow of H_0 is to say that the Poisson bracket

$$\{K, H_0\} := -d\alpha(X_K, X_{H_0}) = dK(X_{H_0})$$

is identically zero. In this case we also say that the Hamiltonians K and H_0 *commute*. In this section we (trivially) adapt Cushman's approach to normal forms of Hamiltonian systems in [10] to show that modulo terms of arbitrary high order every deformation of a star Hamiltonian H_0 with periodic flow is equivalent, via homogeneous symplectic transformations, to a deformation by Hamiltonians that commute with H_0. This, together with Theorem 3.1 and the invariance of the systolic volume under homogeneous symplectic transformations will complete the proof of Theorem 2.1.

Recall that a smooth deformation H_t of a Hamiltonian H_0 is in *normal form up to order* N if

$$H_t = H_0 + tE_1 + \cdots + t^N E_N + o(t^N),$$

where $\{E_i, H_0\} = 0$ $(1 \leq i \leq N)$.

Theorem 4.1. *Let $H_t : T^*M \to [0, \infty)$ be a smooth family of star Hamiltonians defined on the cotangent bundle of a closed manifold M such that the flow of H_0 is periodic. For any order N there exists a smooth deformation $\phi_t^{(N)}$ of the identity map consisting of homogeneous symplectic transformations such that $H_t \circ \phi_t^{(N)}$ is in normal form up to order N.*

In what follows we will denote the space of real-valued functions on T^*M that are homogeneous of degree one and smooth outside the zero section by $\mathcal{H}$. Notice that this space is closed under Poisson brackets. The key step in the proof of Theorem 4.1 is to show that star Hamiltonians with periodic flows give a natural decomposition of $\mathcal{H}$.

Lemma 4.1. *If $H_0 \in \mathcal{H}$ is a star Hamiltonian whose flow is periodic, then the space $\mathcal{H}$ of real-valued functions on T^*M that are homogeneous of degree one and smooth outside the zero section decomposes as a direct sum of the kernel and image of the operator*

$$\mathrm{ad}_{H_0} : \mathcal{H} \longrightarrow \mathcal{H}$$

defined by $\mathrm{ad}_{H_0}(H) = \{H, H_0\}$.

Proof. Let ϕ_t denote the Hamiltonian flow of H_0 and let T be its period. The projection onto $\mathrm{Ker}\,\mathrm{ad}_{H_0}$ associated to this decomposition is the operator that sends a Hamiltonian $H \in \mathcal{H}$ to the averaged Hamiltonian defined by

$$\overline{H}(\xi) = \frac{1}{T}\int_0^T H(\phi_t(\xi))dt.$$

If the averaged Hamiltonian is identically zero, it can be easily checked that $H = \mathrm{ad}_{H_0}(K)$, where

$$K(\xi) = \frac{1}{T}\int_0^T tH(\phi_t(\xi))dt.$$

Therefore, any Hamiltonian H in $\mathcal{H}$ decomposes as a sum $H = \overline{H} + \mathrm{ad}_{H_0}(K)$. □

Proof of Theorem 4.1. Let us first settle the case $N = 1$ where the idea of the proof is most clearly seen. We write $H_t = H_0 + tH_1 + o(t)$ and, applying the previous lemma, decompose H_1 into $\overline{H}_1 + \mathrm{ad}_{H_0}(K_1)$. We denote by $\psi_t^{(1)}$ the Hamiltonian flow corresponding to K_1 and claim that $H_t \circ \psi_t^{(1)}$ is in normal form to order one. Here and it what follows we shall abuse the term *flow*: the map $\psi_t^{(1)}$ may not exist for all values of t, but it will be sufficient for us that it exists for small times (see for Proposition 2.1).

To see that $H_t \circ \psi_t^{(1)}$ is in normal form up to order one, just notice that $H_0 \circ \psi_0^{(1)} = H_0$ and

$$\frac{d}{dt} H_t \circ \psi_t^{(1)}{}_{|t=0} = H_1 + dH_0(X_{K_1}) = H_1 - \mathrm{ad}_{H_0}(K_1) = \overline{H}_1.$$

Therefore, $H_t \circ \psi_t^{(1)} = H_0 + t\overline{H}_1 + o(t)$ is in normal form up to order one.

The construction of the normal form proceeds by induction: assume that the deformation H_t is in normal form up to order $N-1$, write

$$H_t = H_0 + tE_1 + \cdots + t^{N-1}E_{N-1} + t^N H_N + o(t^N),$$

where $\{E_i, H_0\} = 0$ $(1 \leq i \leq N-1)$, and decompose H_N into $\overline{H}_N + \mathrm{ad}_{H_0}(K_N)$. If $\psi_t^{(N)}$ the Hamiltonian flow corresponding to K_N, the deformation $t \mapsto H_t \circ \psi_{t^N}^{(N)}$ is in normal form up to order N. □

Proof of Theorem 2.1. Let $H_t : T^*M \to [0,\infty)$ be a smooth family of star Hamiltonians such that the Hamiltonian flow of H_0 is periodic of period $\mathrm{sys}(S^*_{H_0}M)$. By the previous theorem, for any natural number N there exists a deformation of the identity $\phi_t^{(N)}$ consisting of homogeneous symplectic transformation such that

$$H_t \circ \phi_t^{(N)} = H_0 + tE_1 + \cdots + t^N E_N + o(t^N)$$

and $\{E_i, H_0\} = 0$ $(1 \leq i \leq N-1)$. Remark that the deformation

$$K_t = (H_0 + tE_1 + \cdots + t^N E_N) \circ \phi_t^{(N)^{-1}}$$

is such that $K_t = H_t + o(t^N)$ and, for every fixed value of the deformation parameter, K_t is constant along the orbits of the Hamiltonian flow of

$H_0 \circ {\phi_t^{(N)}}^{-1}$. Theorem 3.1 and the invariance of the systolic volume under homogeneous symplectic transformations allow us to conclude that

$$\mathfrak{S}(S^*_{K_t} M) \geq \mathfrak{S}(S^*_{H_0 \circ {\phi_t^{(N)}}^{-1}} M) = \mathfrak{S}(S^*_{H_0} M).$$

□

5. Final remarks

In this final section we place our main result in context by presenting a short survey of what is known about systolic inequalities on Finsler manifolds and by explaining an alternative approach to infinitesimal rigidity due to Berger [7].

Finsler geometry and systolic inequalities. In retrospect, the first isosystolic inequality ever discovered is Minkowski's celebrated result on the geometry of numbers: *If the (Euclidean) volume of a centrally symmetric convex body $B \subset \mathbb{R}^n$ is at least 2^n times the volume of the fundamental domain of a lattice $\Gamma \subset \mathbb{R}^n$, then B contains a non-zero point of the lattice.* In terms of Finsler geometry this gives:

Theorem 5.1 (Minkowski)**.** *Let T^n be an n-dimensional torus endowed with a flat, reversible Finsler metric. If we denote the Hausdorff measure of T^n by $\mathrm{vol}_{\mathcal{H}}(T^n)$ and the volume of the Euclidean unit ball of dimension n by ϵ_n, we have*

$$\frac{\mathrm{vol}_{\mathcal{H}}(T^n)}{\mathrm{sys}(T^n)^n} \geq \frac{\epsilon_n}{2^n} .$$

To perform the translation between the classical and the Finsler formulation is it enough to remark that (1) if Γ is a lattice in $\mathbb{R}^n$ and $\|\cdot\|$ is the norm whose unit ball is B, then the systole of the flat torus $T^n = (\mathbb{R}^n, \|\cdot\|)/\Gamma$ is the infimum of the norms of all non-zero points in Γ; (2) the Hausdorff measure of T^n is ϵ_n times the ratio of the (Euclidean) volume of a fundamental domain of Γ and the volume of B.

It is not known—even in two dimensions—whether the conclusion of Theorem 5.1 can be extended to all reversible Finsler metrics.

Question. Does every reversible Finsler torus of dimension two satisfy the systolic inequality

$$\frac{\mathrm{vol}_{\mathcal{H}}(T^2)}{\mathrm{sys}(T^2)^2} \geq \frac{\pi}{4} \ ?$$

Remarkably, for the Holmes-Thompson volume the following version has been proved by Sabourau [18].

Theorem 5.2 (Sabourau)**.** *The (Holmes-Thompson) systolic volume of a two-dimensional torus endowed with a reversible Finsler metric is at least $2/\pi$. Equality holds for the flat torus $\mathbb{R}^2/\Gamma$, where the unit ball of the norm* *in $\mathbb{R}^2$ is a parallelogram centered at the origin and, modulo translation and dilation, equal to a fundamental region of Γ.*

Since the Hausdorff measure of a Finsler manifold is always greater than or equal to its Holmes-Thompson volume (see [11]), Sabourau's result implies that

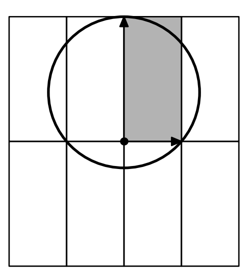
$$\frac{\mathrm{vol}_{\mathcal{H}}(T^2)}{\mathrm{sys}(T^2)^2} \geq \frac{2}{\pi}.$$

The only other sharp systolic inequality for reversible Finsler metrics that is known at present is the following result by Ivanov [14]:

Theorem 5.3 (Ivanov)**.** *The (Holmes-Thompson) systolic volume of the projective plane endowed with a reversible Finsler metric is at least $2/\pi$. Equality holds for Finsler metrics all of whose geodesics are closed and of the same length.*

In Finsler geometry there are many natural definitions of volume (see [20], [4], and [3]) and it is natural to ask what impact this has on the study of systolic and filling inequalities. Given that all reversible Finsler metrics on a closed manifold are bilipschitz to Riemannian metrics with a constant that depends only on the dimension of the manifold, all notions of volume on a reversible Finsler manifold differ by a bounded factor depending only on the dimension. Therefore, coarse questions such as systolic rigidity or freedom of reversible Finsler metric are already answered by the Riemannian theory. On the other hand, the study of sharp isosystolic inequalities or coarse questions for non-reversible metrics require new techniques or exhibit new phenomena. For example, we shall now show that if the Busemann-Hausdorff definition of volume is used, the systolic volume of Finsler tori can be arbitrarily close to zero.

Consider the family of non-reversible norms $\|\cdot\|_\lambda$ on $\mathbb{R}^2$ whose unit disc is the Euclidean disc of radius one centered at the point $(0,\lambda)$ $(0 \leq \lambda < 1)$. For a given $\lambda \in [0,1)$, let $\Gamma_\lambda \subset \mathbb{R}^2$ be the lattice spanned by the vectors $(\sqrt{1-\lambda^2}, 0)$ and $(0, 1+\lambda)$, and let T_λ be the flat torus $(\mathbb{R}^2, \|\cdot\|_\lambda)/\Gamma_\lambda$.



The lattice Γ_λ inside the non-reversible normed plane $(\mathbb{R}^2, \|\cdot\|_\lambda)$.

Proposition 5.1. *If we use the Busemann-Hausdorff definition of volume, the systolic volume of the torus T_λ is equal to $(1+\lambda)\sqrt{1-\lambda^2}$ and hence tends to zero as λ tends to 1. On the other hand, if the Holmes-Thompson definition is used, the systolic volume of T_λ is $(1-\lambda)^{-1}$ and hence tends to infinity as λ tends to 1.*

Proof. Remark that the lattice Γ_λ has been chosen so that $\mathrm{sys}(T_\lambda) = 1$. It remains for us to compute the Busemann-Hausdorff and the HolmesThompson volume of the parallelogram P_λ spanned by the vectors $(\sqrt{1-\lambda^2}, 0)$ and $(0, 1+\lambda)$ in the non-reversible normed plane $(\mathbb{R}^2, \|\cdot\|_\lambda)$.

By definition, the Busemann-Hausdorff area density on a (non-reversible) normed plane is the multiple of the Lebesgue measure that assigns the value π to the unit disc. Therefore, by construction, the Busemann-Hausdorff area of the parallelogram P_λ in $(\mathbb{R}^2, \|\cdot\|_\lambda)$ equals its Euclidean area:

$$(1+\lambda)\sqrt{1-\lambda^2}.$$

Notice that even if the systolic volume of T_λ eventually tends to zero as λ tends to 1, it starts by increasing for small positive values of λ.

By definition, the Holmes-Thompson area density of a (non-reversible) normed plane with unit disc D is the multiple of the Lebesgue measure that assigns D the quantity $1/\pi$ times the symplectic volume of $D \times D^*$ in the space $\mathbb{R}^2 \times \mathbb{R}^{2*}$. If we use the Euclidean area on $\mathbb{R}^2$ to perform the computation this is tantamount to saying that the Holmes-Thompson area of P_λ equals $1/\pi$ times the product of the Euclidean areas of P_λ and the dual unit disc $D^*_\lambda \subset (\mathbb{R}^{2*}, \|\cdot\|^*_\lambda)$.

In order to compute the Euclidean area of D^*_λ, notice that the dual norm $\|\cdot\|^*_\lambda$ is the support function of the unit disc D_λ, which is just the Euclidean unit disc translated by the vector $(0, \lambda)$, and thus it is given by the formula

$$\|(\xi_1, \xi_2)\|^*_\lambda = \sqrt{\xi_1^2 + \xi_2^2} + \lambda\xi_2 \ .$$

Therefore, the dual unit disc is the ellipse

$$(1-\lambda^2)\,\xi_1^2 + (1-\lambda^2)^2\,(\xi_2 + (\lambda/(1-\lambda^2))^2 = 1$$

whose volume equals $\pi(1-\lambda^2)^{-3/2}$. Summing up, the (Holmes-Thompson) systolic volume of T_λ is

$$(1+\lambda)\,\sqrt{1-\lambda^2}\,(1-\lambda^2)^{-3/2} = (1-\lambda)^{-1} \ .$$

□

It is possible that examples of non-reversible Finsler metrics with arbitrarily small Busemann-Hausdorff systolic volume exist on any closed manifold.

Question. Given a closed n-dimensional manifold M and a positive number ϵ, is it always possible to find a non-reversible Finsler metric on M for which

$$\frac{\mathrm{vol}_{\mathcal{H}}(M, F)}{\mathrm{sys}(M, F)^n} \leq \epsilon \ ?$$

On the other hand, the Holmes-Thompson volume penalizes lack of symmetry and it may be that some results of systolic rigidity extend to non-reversible Finsler metrics if this notion of volume is adopted.

Question. Is the (Holmes-Thompson) systolic volume of a non-reversible Finsler metric on the two-dimensional torus greater than some constant $\delta > 0$ independent of the metric?

Alternative definition of infinitesimal rigidity. In order to clarify the difference between our approach to infinitesimal systolic inequalities and that of Berger [7], we propose the following definitions:

Definition 5.1. Let f be a function defined on a manifold M. A point $x \in M$ is a *pseudo-minimum of f to order k* if f is continuous at x and for every smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x$ there exists another smooth curve σ also defined in a neighborhood of $t = 0$ and such that (1) both curves agree to order k at $t = 0$; (2) $f(\sigma(t)) \geq f(x)$ for all values of t in the domain of σ.

Definition 5.2. Let f be a function defined on a manifold M. A point $x \in M$ is a *pseudo-critical* point of f if this function is continuous at x and there exists a differentiable function h defined in a neighborhood of x and such that (1) $h(x) = f(x)$; (2) $h \leq f$; (3) x is a critical point of h.

Berger shows that *the canonical metric on $\mathbb{R}P^n$ is a pseudo-critical point for the systolic volume as a function on the space of Riemannian metrics on $\mathbb{R}P^n$.* On the other hand, we show that a closed Finsler manifold (M, F) all of whose geodesics are closed and of the same length is a pseudo-minimum to all orders for the systolic volume as a function on the space of Finsler metrics on M.

Unfortunately, the notions are not comparable and a pseudo-minimum is not necessarily a pseudo-critical point and vice-versa. Our choice of viewpoint is partly explained by the fact that Berger's proof does not extend to simply-connected manifolds nor does it extend to Finsler deformations of the canonical metric of $\mathbb{R}P^n$.

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We present some results concerning the Morse Theory of the energy function on the free loop space of the three sphere for metrics all of whose geodesics are closed. We also explain how these results relate to the Berger conjecture in dimension three. Keywords. Berger conjecture Morse theory Manifolds all of whose geodesics are closed Three sphere. The results presented in this article are part of my doctoral thesis at the University of Pennsylvania. This is a preview of subscription content, log in to check access. Preview. Unable to display preview. The Zoll surfaces have the property that all of their geodesics are closed. If one futher stipulates that all geodesics are also simple, i.e., non-self-intersecting, does this leave only the sphere? Apologies for the simplicity of this question, but I am not finding an answer in the literature, and I suspect many just know this off the top of their head. Thanks! riemannian-geometry.